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Algebraic Inequalities: New Vistas

Titu Andreescu
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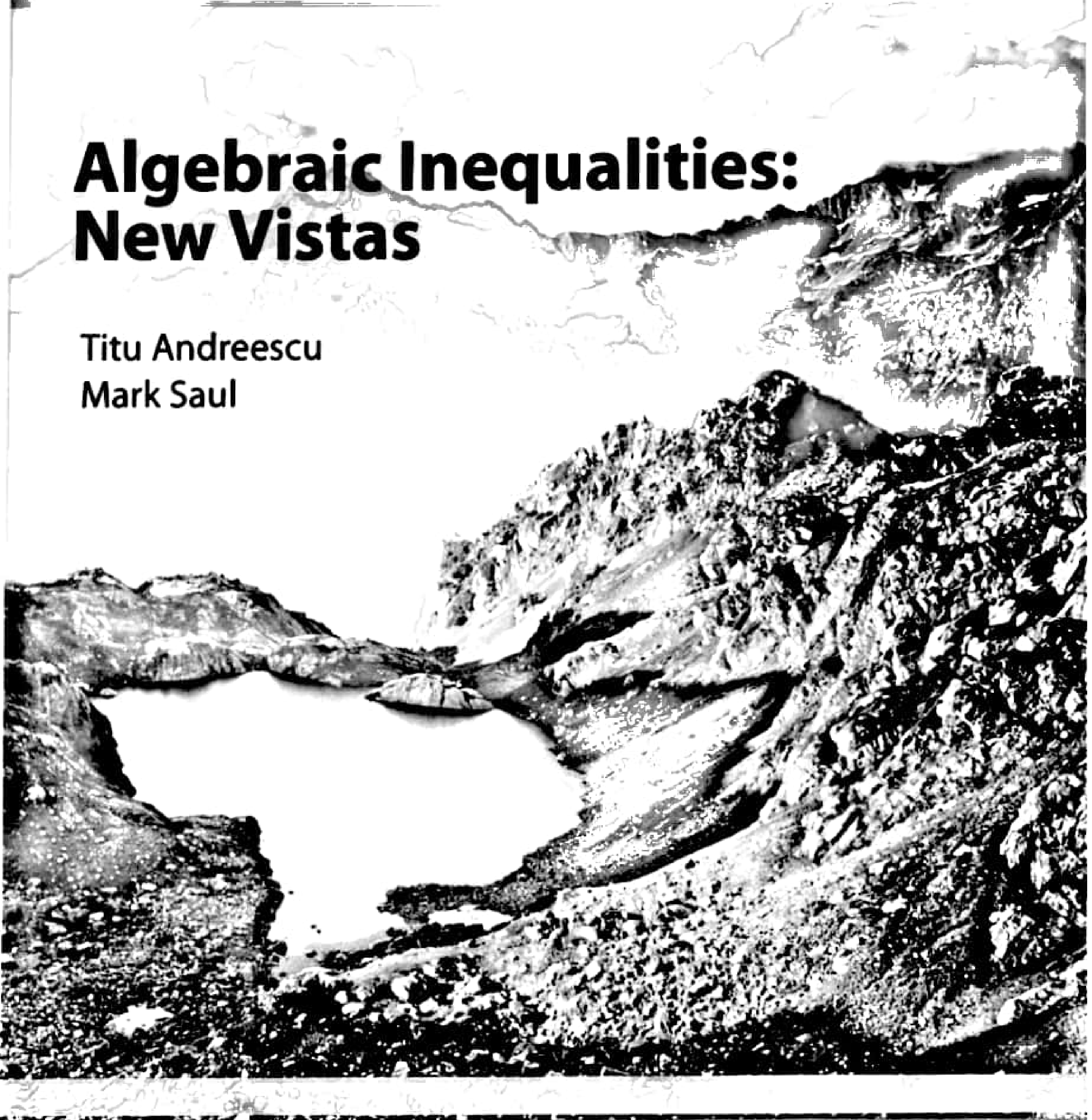

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Titu Andreescu
Mark Saul

Introduction

This book is about algebraic inequalities. We have chosen this topic because we can start with almost no background, and build meaningful sets of problems.

At the outset, nothing is required of the reader beyond an elementary knowledge of the rational numbers. Later, we require some algebra, and later still more algebra, up to and including the solution of quadratic equations. For all but a very few problems, no more than intermediate algebra is required as background.

The exposition is not exactly linear. We have taken the opportunity to link our central topic with others in mathematics. Sometimes these topics relate to much more than algebraic inequalities, but arise naturally as we discuss our topic. An example is the long digression on algebraic symmetry in the middle of the book. The topic creeps into the early chapters, and is addressed directly in the middle chapters.

There are also 'secret' pathways through the book. Each chapter has a subtext, a theme which prepares the student for learning other mathematical topics, concepts, or habits of mind. The early chapters on the AM/GM inequality, for example, show how very simple observations can be leveraged to yield useful and interesting results. The later chapters give examples of how one can generalize a mathematical statement. The chapter on the Cauchy-Schwarz inequality provides an introduction to vectors as mathematical objects. And there are many other secret pathways that we hope the reader will discover—and follow.

There is another level to this book. If you are reading this, you probably enjoy solving math problems. It is a joy to find a good solution to a problem, and perhaps the greatest joy comes from the most difficult problem. But if you have not solved difficult math problems, how do you learn this skill?

In regular math classes, or regular textbooks, you learn to read mathematics. You learn about the objects of mathematics and acquire tools for working with them. The mathematical tasks set in most textbooks offer exercise, but not challenge.

On the other hand, contest problems, or problems posed in journals, are often very difficult. Many people are successful in school mathematics,

but don't have the problem solving skills to approach even a relatively easy problem that is not part of the structure they have learned in school.

How do you learn to solve problems? How do you get there from here?

This book attempts an answer. Each chapter begins with problems requiring only a little thought, but still—we hope—evoking thoughts that the solver had not yet had. Each chapter then leads the reader through more and more difficult problems. The final problems are olympiad level and sometimes require advanced knowledge.

We have linked the problems, so that each contains a hint, or a preparatory result, for the next. We have given hints—fewer as the difficulty of the problem progresses. We have also written solutions which peer forward to later problems and back to earlier ones, in the hopes that that the reader learns something valuable from each solution.

Thus it is important to read our solutions, even if you have your own. And even if your own is better.

We think of our work as marking a trail for the student, a trail which starts out flat and smooth, leads through tougher and tougher terrain, and ends with a scramble to the summit. But you need not get all the way up to enjoy the trip. We have striven to provide vistas of learning at each turn in the path.

The hike may be strenuous, but the vistas are worth it. We hope you enjoy the climb!

Titu Andreescu
Mark Saul

Chapter 0

Some Introductory Problems

Sometimes we take for granted the most important things. For example, it is 'obvious', but very important, that real numbers can be compared to each other. Of two real numbers, one must be greater than the other—unless they are equal. This is a basic property both of the real numbers and of the inequality relation on them. The arithmetic problems below are intended to bring out a few more important, easy, but subtle properties of arithmetic inequalities. Please do not use a calculator for these problems.

0.1. A man was collecting from an audience for a certain charity. He said, "Look at the money in your pocket. You can certainly donate $\frac{1}{10}$ of that money to us. But if you can't afford to give $\frac{1}{10}$, maybe you can afford to give $\frac{1}{9}$ or $\frac{1}{8}$." What comment do you have on this scene?

0.2. Which is larger: (A) $\frac{9999}{10000}$ or (B) $\frac{10000}{10001}$?

0.3. Which is larger: (A) $\frac{90046}{90049}$ or (B) $\frac{90047}{90050}$?

0.4. Which is larger: (A) $\frac{1}{2+\frac{3}{7}}$ or (B) $\frac{1}{2+\frac{4}{7}}$?

0.5. Which is larger: (A) $\dfrac{1}{2+\dfrac{3}{6-\frac{4}{11}}}$ or (B) $\dfrac{1}{2+\dfrac{3}{6-\frac{5}{11}}}$?

0.6. A remark attributed to Joseph Stalin: "That tendency is not just a negative quantity. It is a negative quantity squared." Comment?

0.7. If a, b, and c are positive real numbers, which is larger:
(A) the average (arithmetic mean) of a, b, and c,
or
(B) the average (arithmetic mean) of a^2, b^2, and c^2?
Hint: Can you always answer this question?

0.8. If c and d are positive numbers, which is larger:
(A) $7c + 9d$ or (B) $9(c + d)$?

Two questions with tricks in them:

0.9. If e and f are real numbers, and $e^2 - 1 > f^2 - 1$, then which is larger: (A) e or (B) f?

0.10. If g and h are real numbers, and $g^4 - 1 > h^4 - 1$, then which is larger: (A) g^2 or (B) h^2?

How Do Inequalities Behave?

Most students think that inequalities are just like equations: "Whatever you do to one side, you can do to the other". This is not quite true, even for equations. But it is "less true" for inequalities.

Let us make some more precise statements about how inequalities differ from equations.

Statement 1. For real numbers a, b, if $a > b$ then $-a < -b$.

Check that this statement is true, whether a and b are positive, negative, or zero.

Statement 2. For positive real numbers a, b, if $a > b$ then $\frac{1}{a} < \frac{1}{b}$.

What can you say if either of the numbers a, b is negative?

Statement 3. If a, b, c, d are positive real numbers and $\frac{a}{b} > \frac{c}{d}$, then $ad > bc$.

This property of inequalities is not really very different from the corresponding property for equations. But it is sometimes overlooked, so we list it separately here. Can you prove statement 3 from statements 1 and 2?

There is one more important statement about inequalities:

Statement 4. (Transitivity Property of Inequality) For real numbers a, b, c, if $a > b$ and $b > c$, then $a > c$.

Note that if we replace the ">" sign with "=", the statement is still true. So this is not exactly a difference between equations and inequalities. But, as we will see, the transitive property becomes much more vivid in working with inequalities than in solving equations.

We will not give a formal treatment of the algebra of inequalities here. An axiomatic description of an ordered field can be found in most books on abstract algebra. We simply want to point out how the algebra of inequalities differs from what we may be used to in working with equations.

Solutions

0.1. A man was collecting from an audience for a certain charity. He said, "Look at the money in your pocket. You can certainly donate $\frac{1}{10}$ of that money to us. But if you can't afford to give $\frac{1}{10}$, maybe you can afford to give $\frac{1}{9}$ or $\frac{1}{8}$." What comment do you have on this scene?

Solution. The man thinks that because $10 > 9 > 8$, then $\frac{1}{10}, \frac{1}{9}, \frac{1}{8}$ are in the same order.

But in fact they are in the opposite order. This is what statement 2 above asserts.

0.2. Which is larger: (A) $\frac{9999}{10000}$ or (B) $\frac{10000}{10001}$?

Solution. An easy way to think of this problem is to ask which fraction is closer to 1. We have:

$$\frac{9999}{10000} = 1 - \frac{1}{10000},$$

$$\frac{10000}{10001} = 1 - \frac{1}{10001},$$

and we know that $\frac{1}{10000} > \frac{1}{10001}$. So, in the first case, you must subtract a larger fraction from 1 than in the second case, to get the fraction on the left.

That is, both fractions are less than 1, but the first is farther from 1 than the second, so the first is smaller.

0.3. Which is larger: (A) $\frac{90046}{90049}$ or (B) $\frac{90047}{90050}$?

Solution. We can proceed here in the same way as in Problem 0.2:

$$\frac{90046}{90049} = 1 - \frac{3}{90049},$$

$$\frac{90047}{90050} = 1 - \frac{3}{90050},$$

and $\frac{3}{90049} > \frac{3}{90050}$, so we see that fraction (B) is larger than fraction (A).

0.4. Which is larger: (A) $\frac{1}{2+\frac{3}{7}}$ or (B) $\frac{1}{2+\frac{4}{7}}$?

Solution 1. We must work from the "bottom up".

Fraction (A) is equal to $\frac{1}{(\frac{17}{7})} = \frac{7}{17}$.

Fraction (B) is equal to $\frac{1}{(\frac{18}{7})} = \frac{7}{18}$, and, as in Problem 0.1, the first fraction is larger than the second. Thus fraction (A) is larger than fraction (B).

Solution 2. Just look to see that the denominator in (B) (that is, $2 + \frac{4}{7}$) is larger than denominator in (A) (that is, $2 + \frac{3}{7}$), so the first fraction is larger than the second.

0.5. Which is larger: (A) $\frac{1}{2+\frac{3}{6-\frac{4}{11}}}$ or (B) $\frac{1}{2+\frac{3}{6-\frac{5}{11}}}$?

Solution 1. Again, work from the bottom up:

$$\frac{1}{2+\dfrac{3}{6-\frac{4}{11}}} = \frac{1}{2+\dfrac{3}{\left(\frac{62}{11}\right)}} = \frac{1}{2+\frac{33}{62}} = \frac{1}{\left(\frac{157}{62}\right)} = \frac{62}{157},$$

$$\frac{1}{2+\dfrac{3}{6-\frac{5}{11}}} = \frac{1}{2+\dfrac{3}{\left(\frac{61}{11}\right)}} = \frac{1}{2+\frac{33}{61}} = \frac{1}{\left(\frac{157}{61}\right)} = \frac{61}{157}.$$

Hence fraction (A) is greater than fraction (B).

Solution 2. This can be done (pretty easily!) by "just looking" at denominators on the way up the two fractions. But it's hard to write down the thought process. Try it yourself.

0.6. A remark attributed to Joseph Stalin: "That tendency is not just a negative quantity. It is a negative quantity squared." Comment?

Solution. Well, this is not exactly a "solution". But note that when you square a negative quantity, you certainly get a positive quantity. Stalin was trying to say that the new quantity is worse (more negative) than the original negative quantity, and got mixed up.

It's not clear that this is an authentic quote. The sources we have found so far are all written by mathematicians.

Can you prove, from our statements (1) and (2), that if $a < 0$, then $a^2 > 0$?

0.7. If a, b, and c are positive real numbers, which is larger:
(A) the average (arithmetic mean) of a, b, and c,
or
(B) the average (arithmetic mean) of a^2, b^2, and c^2?
Hint: Can you always answer this question?

Solution. The average of three bigger numbers is certainly larger than the average of three smaller numbers. So it looks like the average of a^2, b^2, c^2 should be larger than the average of a, b, c.

But in fact squaring does not always make a number larger! Not even a positive number: if $0 < x < 1$, then $0 < x^2 < x < 1$. So you can't tell which of these two averages is the larger.

Can you prove the statement we just made (about x), from our statements (1) and (2)?

0.8. If c and d are positive numbers, which is larger:
(A) $7c + 9d$ or (B) $9(c + d)$?

Solution. We have $9(c + d) = 9c + 9d = 7c + 9d + 2c$. So we have to add something positive to expression (A) to get to expression (B). Hence (B) must be the larger.

Is the same thing true if c and d are negative numbers?

0.9. If e and f are real numbers, and $e^2 - 1 > f^2 - 1$, then which is larger: (A) e or (B) f?

Solution. If $e^2 - 1 > f^2 - 1$, then we know $e^2 > f^2$. But does that mean that $e > f$?

Certainly not. Find some examples for yourself.

0.10. If g and h are real numbers, and $g^4 - 1 > h^4 - 1$, then which is larger: (A) g^2 or (B) h^2?

Solution. If $g^4 - 1 > h^4 - 1$, then we know $g^4 > h^4$. But does that mean that $g^2 > h^2$?

Yes, it does! It does because all the quantities we are comparing are squares ($g^4 = (g^2)^2$ counts as a square), so we don't have those annoying negative numbers to worry about.

Chapter 1

Squares Are Never Negative

Theorem 1.1. *For any real number N, we have $N^2 \geq 0$, with equality if and only if $N = 0$.*

This seems simple. Math students know that the square of a real number cannot be negative. There's hardly anything here to prove.

But look at what we can do with this result.

Example 1.1. Show that for any two real numbers a, b, we have $a^2 + b^2 \geq 2ab$.

Solution. We know that $(a-b)^2 \geq 0$ (because a square is never negative). So $a^2 - 2ab + b^2 \geq 0$, and $a^2 + b^2 \geq 2ab$. Isn't this now "obvious"? Of course! "Obvious" really just means "I figured it out".

But how could we think of starting with $(a-b)^2$? Well, we could have worked backwards. We want $a^2 + b^2 \geq 2ab$, which is equivalent to $a^2 - 2ab + b^2 \geq 0$. The expression on the left looks familiar, and factoring it reduces the original statement to an "obvious" one: $(a-b)^2 \geq 0$. Then we write down the proof in the more logical order, as in the first sentence of this solution.

When can $a^2 + b^2 = 2ab$? Again, we can follow the logic "backwards". We are asking when $(a-b)^2 = 0$, and this can only occur when $a = b$.

In general, whenever we have an inequality, we want to know when equality holds. This will become important in later problems.

Problems

1.1. Show that for any two positive numbers c, d, we have $c + d \geq 2\sqrt{cd}$.

1.2. For any real numbers a and b, prove that

$$12(a^2 - ab + b^2) \geq 6(a^2 + b^2) \geq 4(a^2 + ab + b^2) \geq 3(a+b)^2 \geq 12ab.$$

1.3. For $a \geq 0$, prove that $a + 1 \geq 2\sqrt{a}$.

1.4. **a.** For any three real numbers a, b, c, show that $a^2 + 2b^2 + c^2 \geq 2ab + 2bc$.

b. For any three real numbers a, b, c, show that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

c. State and prove an analogous result for four real numbers a, b, c, d.

1.5. If a is a positive number, find the minimum possible value of the expression $a - 2\sqrt{a}$.

1.6. For any real number b, find the minimum possible value of $b^2 - 6b$.
1.7. For any real number c, find the minimum possible value of $c^2 - 8c + 7$.

♦

Example 1.2. For any two positive numbers x and y, show that

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

Solution. Clearing fractions, we have $x^2 + y^2 \geq 2xy$, which is true from Example 1.1.

But why must we stipulate that x and y be positive? Well, the statement is certainly wrong if we don't. Let $x = 1$, $y = -1$. Then $\frac{x}{y} + \frac{y}{x} = -2$, which is certainly less than 2. Where did the proof go wrong?

Recall (from Chapter 0) that inequalities do not behave just like equalities. If $P > Q$, then $kP > kQ$ if k is positive, but $kP < kQ$ if k is negative. When we "cleared fractions" above, we multiplied both sides of the inequality by xy. So we had to be sure that this quantity is positive. The condition that x and y both be positive assures us of this.

In this case, we can go a bit further: we can say that if $xy > 0$ (that is, if x and y have the same sign), then $\frac{x}{y} + \frac{y}{x} \geq 2$.

♦

1.8. For any positive number a, find the minimum value of the expression

$$a + \frac{1}{a}.$$

1.9. For any positive number c, show that $c + \frac{2}{c} \geq 2\sqrt{2}$.

1.10. For any positive number d, what is the minimum value of the expression

$$d + \frac{3}{d}?$$

1.11. If $a > b > 0$, show that

$$\frac{1}{8} \cdot \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8} \cdot \frac{(a-b)^2}{b}.$$

Later in this volume, we will see that this result puts bounds on the difference between the arithmetic and geometric mean of two numbers.

1.12. If a, b, c, d are four real numbers such that $ab \geq 0$ and $cd \geq 0$, show that

$$ad + bc \geq 2\sqrt{abcd}.$$

Can the inequality be made "sharper"? That is, can the number 2 on the right-hand side be replaced by any larger number?

1.13. For $a, b, c, d \geq 0$, show that $\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}$.

1.14. If a, b, c, d are all positive, and $\frac{a}{b} \leq \frac{c}{d}$, show that $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$.

1.15. Suppose we have n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$. If m and M are the smallest and largest of these fractions (respectively), show that

$$m \le \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \le M.$$

This problem generalizes Problem 1.14.

1.16. Prove that for all real numbers x,

$$4(x^3 + x + 1) \le (x^2 + 1)(x^2 + 5).$$

◆

The next few problems involve the expressions $(a+b)^3$, $a^3 + b^3$, and $a^3 - b^3$. Recall that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

◆

1.17. For $a, b \ge 0$, show that $a^2 - ab + b^2 \ge ab$.

1.18. Make sure you know how to factor $a^3 + b^3$. Then show that for $a, b \ge 0$,

$$a^3 + b^3 \ge ab(a+b).$$

1.19. Show that for non-negative numbers a, b, we have

$$\frac{a^3 + b^3}{2} \ge \left(\frac{a+b}{2}\right)^3.$$

1.20. For any non-negative number a, prove that $1 - \frac{a}{2} \le \frac{1}{1 + a^2}$.

Hint: If you run out of ideas, try factoring.

1.21. Let a and b be positive real numbers. Prove that

$$\left|\frac{a^3 - b^3}{a^3 + b^3}\right| \le 3\left|\frac{a-b}{a+b}\right|.$$

1.22. Compute the product $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$.

1.23. Show that the rightmost factor in the product of Problem 1.22 can be written as $\frac{1}{2}\left((x-y)^2 + (y-z)^2 + (z-x)^2\right)$.

1.24. If x, y, z are real numbers, show that the expression

$$x^2 + y^2 + z^2 - xy - yz - zx$$

is zero if and only if $x = y = z$.

1.25. In a triangle whose sides have lengths a, b, c, with $a < b < c$, show that

$$a^3 + b^3 + c^3 + 3abc > 2c^3.$$

1.26. (This problem requires some knowledge of complex numbers.) If we have complex numbers v, w, z such that $v^2+w^2+z^2-vw-wz-zv=0$, show that either $v=w=z$ or the points represented by v, w, z in the complex plane form an equilateral triangle.

Solutions

1.1. Show that for any two positive numbers c, d, we have $c+d \geq 2\sqrt{cd}$.

Solution. This falls to a typical algebraic "trick". In Example 1.1, let $a=\sqrt{c}$, and let $b=\sqrt{d}$, and the statement falls out.
Equality holds if and only if $c=d$.
Why did we have to say that c and d were positive numbers here?
This innocent-looking statement will soon become a powerful tool. (If you're impatient to find out how, look for *Arithmetic-Geometric Mean Inequality* later in this book.)

1.2. For any real numbers a and b, prove that

$$12(a^2-ab+b^2) \geq 6(a^2+b^2) \geq 4(a^2+ab+b^2) \geq 3(a+b)^2 \geq 12ab.$$

Solution. Each link in this chain opens to routine algebraic manipulation. We offer the first two as examples:

(i) We need to show that

$$12(a^2-ab+b^2) \geq 6(a^2+b^2),$$

which we can write as:

$$12(a^2-ab+b^2)-6(a^2+b^2) \geq 0.$$

We have:

$$\begin{aligned}12a^2-12ab+12b^2-6a^2-6b^2 &= 6a^2-12ab+6b^2\\ &= 6(a^2-2ab+b^2)\\ &= 6(a-b)^2,\end{aligned}$$

which is certainly greater than or equal to 0. Equality holds just when $a=b$.

(ii) We have:

$$\begin{aligned}6(a^2+b^2) &\geq 4(a^2+ab+b^2)\\ 6a^2+6b^2 &\geq 4a^2+4ab+4b^2\\ 2a^2-4ab+2b^2 &\geq 0\\ 2(a-b)^2 &\geq 0,\end{aligned}$$

which is certainly true, with equality exactly when $a=b$.

1.3. For $a \geq 0$, prove that $a+1 \geq 2\sqrt{a}$.

Solution. We can write the given inequality as

$$a-2\sqrt{a}+1 \geq 0.$$

The form of the expression on the left is reminiscent of a perfect square, especially given the title of this set of problems. And in fact this inequality is equivalent to

$$(\sqrt{a}-1)^2 \geq 0,$$

which is just the theme we are varying here. Equality holds if and only if $a=1$.

1.4. **a.** For any three real numbers a, b, c, show that $a^2+2b^2+c^2 \geq 2ab+2bc$.

Solution. Following the theme of this chapter, we can try to show that this is the consequence of the fact that a square is never negative. So let's write it as a statement that something is never negative. The required inequality is equivalent to

$$a^2+2b^2+c^2-2ab-2bc \geq 0.$$

And now we can create squares, by factoring. Notice that b^2 is treated differently from a^2 or c^2. That's what makes the problem 'ugly'. So there must be a reason for this.
Indeed, we can make the problem prettier by writing it as:

$$a^2+b^2+b^2+c^2-2ab-2bc \geq 0,$$

or

$$a^2-2ab+b^2+b^2-2bc+c^2 \geq 0,$$

or

$$(a-b)^2+(b-c)^2 \geq 0,$$

which is certainly true. Of course, we must check to see that the reasoning is reversible. Happily, it is.
Did you see the squares coming?

b. For any three real numbers a, b, c, show that $a^2+b^2+c^2 \geq ab+bc+ca$.

Solution. Comparing this problem with Problem 1.4a might give us an idea. Rewrite the required inequality as:

$$a^2+b^2+c^2-ab-bc-ca \geq 0.$$

Now what? We want something like $(a-b)^2 = a^2-2ab+b^2$, etc. We are lacking certain squares, but even more conspicuously, we are lacking double copies of ab, bc, ca. So let's supply them, by writing the inequality as:

$$2a^2+2b^2+2c^2-2ab-2bc-2ca \geq 0,$$

which is still equivalent. And let's try to 'cannibalize' the expression for squares:

$$a^2+a^2-2ab+b^2+b^2+2c^2-2bc-2ca \geq 0.$$

The form of the expression on the left is reminiscent of a perfect square, especially given the title of this set of problems. And in fact this inequality is equivalent to

$$(\sqrt{a}-1)^2 \geq 0,$$

which is just the theme we are varying here. Equality holds if and only if $a = 1$.

1.4. **a.** For any three real numbers a, b, c, show that $a^2 + 2b^2 + c^2 \geq 2ab + 2bc$.

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Indeed, we can make the problem prettier by writing it as:

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or

$$a^2 - 2ab + b^2 + b^2 - 2bc + c^2 \geq 0,$$

or

$$(a-b)^2 + (b-c)^2 \geq 0,$$

which is certainly true. Of course, we must check to see that the reasoning is reversible. Happily, it is.

Did you see the squares coming?

b. For any three real numbers a, b, c, show that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Solution. Comparing this problem with Problem 1.4a might give us an idea. Rewrite the required inequality as:

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 0.$$

Now what? We want something like $(a-b)^2 = a^2 - 2ab + b^2$, etc. We are lacking certain squares, but even more conspicuously, we are lacking double copies of ab, bc, ca. So let's supply them, by writing the inequality as:

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \geq 0,$$

which is still equivalent. And let's try to 'cannibalize' the expression for squares:

$$a^2 + a^2 - 2ab + b^2 + b^2 + 2c^2 - 2bc - 2ca \geq 0.$$

We have an 'extra' copy of a^2, but let's keep going:

$$a^2 + (a-b)^2 + b^2 - 2bc + c^2 + c^2 - 2ca \geq 0,$$
$$a^2 + (a-b)^2 + (b-c)^2 + c^2 - 2ca \geq 0,$$

and we're home. The 'extra' pieces make up yet another square:

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0.$$

Many students try to start this problem by computing $(a-b-c)^2$, or some such trinomial square. The problem is that the result we want is *symmetric* in a, b, and c, but the trinomial we are squaring is not. This concept will be explored in some depth later on in this volume.

c. State and prove an analogous result for four real numbers a, b, c, d.
Solution. Having worked on the solution to Problem 1.4b, we can start from the end, and see what inequality results. The natural way to do this is to write

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 \geq 0;$$
$$a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2cd + d^2 + d^2 - 2da + a^2 \geq 0,$$

and eventually

$$a^2 + b^2 + c^2 + d^2 \geq ab + bc + cd + da.$$

Can you generalize for n real numbers, $a_1, a_2, \ldots, a_n$? As an exercise in subscripts, try writing down the result, without necessarily deriving. The derivation will not be anything new.
Problems like this one are of course not well-defined. Perhaps you may think of another generalization, even more clever than this one.

1.5. If a is a positive number, find the minimum possible value of the expression $a - 2\sqrt{a}$.

Solution. This is just another way of asking the question of Problem 1.3. From that problem, we have $a - 2\sqrt{a} \geq -1$, with equality when $a = 1$. Hence the minimum possible value of the expression is -1.
Inequalities are often used to solve problems involving maxima and minima. Note that without having solved Problem 1.3 we would have to think much harder about how to solve this problem. But see the note to the next problem.

1.6. For any real number b, find the minimum possible value of $b^2 - 6b$.

Solution. There are many ways to solve this problem. We present a proof using the theme of this chapter.
We want to relate the given expression to the square of a real number, and in fact, we have the "makings" of a trinomial square. If we just add 9 to this expression, we have:

$$b^2 - 6b + 9 = (b-3)^2 \geq 0.$$

So $b^2 - 6b \geq -9$, with equality when $b = 3$.

That is, the minimum value of the given expression is -9, and is achieved when $b = 3$.

This method may seem artificial, but in fact, it is another method that generalizes powerfully.

1.7. For any real number c, find the minimum possible value of $c^2 - 8c + 7$.

Solution. We can again look for a perfect square, hiding behind the expression we are given:

$$c^2 - 8c + 7 = c^2 - 8c + 16 - 9 = (c-4)^2 - 9 \geq 0 - 9 = -9,$$

with equality just when $c = 4$.

How did we decide to express 7 as $16 - 9$? If your hindsight doesn't show you this, look up the method of *completing the square* in any intermediate algebra text.

1.8. For any positive number a, find the minimum value of the expression

$$a + \frac{1}{a}.$$

Solution 1. Taking a hint from Example 1.2, we let $x = a$, $y = 1$ to get

$$\frac{a}{1} + \frac{1}{a} \geq 2,$$

with equality when $a = 1$.

What is the corresponding result if a is a negative number?

Solution 2. We can proceed directly as in Problem 1.1, then use the fact that $(A - B)^2 \geq 0$ for any real numbers A, B.

Here we let $A = \sqrt{a}$, $B = \dfrac{1}{\sqrt{a}}$, and expand. We get that

$$a + \frac{1}{a} - 2 \geq 0,$$

which leads to the same result as in the first solution.

1.9. For any positive number c, show that $c + \dfrac{2}{c} \geq 2\sqrt{2}$.

Solution. We cannot use the result of Example 1.2 directly. But we can "imitate" the proof that got us its result. That is, clearing fractions, we have $c^2 + 2 \geq 2c\sqrt{2}$. Letting $a = c$, $b = \sqrt{2}$ in Example 1.1, we have our result.

The condition for equality in Example 1.1 implies that equality holds here when $c = \sqrt{2}$. Check to make sure that this is correct.

Later we will see another way to think of this problem.

1.10. For any positive number d, what is the minimal value of the expression

$$d + \frac{3}{d}?$$

Solution. Looking at the solution to Problem 1.9, we might guess that the minimum value is $2\sqrt{3}$.

But pretend we didn't guess this. Let the minimum value be some

number k. Then we have $d + \frac{3}{d} \geq k$, or $d^2 + 3 \geq kd$. If we match this with Example 1.1, we can try letting $a = d$, $b = \sqrt{3}$. Then Example 1.1 tells us that $d^2 + 3 \geq 2d\sqrt{3}$. Since $d > 0$, we can write this as

$$d + \frac{3}{d} \geq 2\sqrt{3}.$$

So we can feel more secure that the minimum value of the given expression is $2\sqrt{3}$. But is this true? Is this minimum actually achieved? The answer is yes, from the equality condition of Example 1.1. If $d = \sqrt{3}$, the minimum we have "predicted" is achieved.

See how important it is to determine the case for equality?

1.11. If $a > b > 0$, show that

$$\frac{1}{8} \cdot \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8} \cdot \frac{(a-b)^2}{b}.$$

Later in this volume, we will see that this result puts bounds on the difference between the arithmetic and geometric mean of two numbers.

Solution. We give a proof for the inequality on the left. The one on the right is obtained analogously.

We want:

$$\frac{1}{8} \cdot \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab}.$$

Direct computation will be a mess, unless we have some insight into where we are going. So look at the right-hand side. Does it look familiar? Well, twice this expression is $a + b - 2\sqrt{ab} = \left(\sqrt{a} - \sqrt{b}\right)^2$, so we can rewrite our inequality as:

$$\frac{1}{4} \cdot \frac{(a-b)^2}{a} \leq \left(\sqrt{a} - \sqrt{b}\right)^2,$$

$$\frac{1}{4a}\left(\sqrt{a} + \sqrt{b}\right)^2 \left(\sqrt{a} - \sqrt{b}\right)^2 \leq \left(\sqrt{a} - \sqrt{b}\right)^2,$$

$$\frac{1}{4a}\left(\sqrt{a} + \sqrt{b}\right)^2 \leq 1,$$

$$\left(\sqrt{a} + \sqrt{b}\right)^2 \leq 4a,$$

and since everything is positive, we can write this as $\sqrt{a} + \sqrt{b} \leq 2\sqrt{a}$, or $\sqrt{b} \leq \sqrt{a}$, which is certainly true for $0 \leq b \leq a$.

Must a and b be non-negative?

1.12. If a, b, c, d are four real numbers such that $ab \geq 0$ and $cd \geq 0$, show that

$$ad + bc \geq 2\sqrt{abcd}.$$

Can the inequality be made "sharper"? That is, can the number 2 on the right-hand side be replaced by any larger number?

Solution. The result follows from the fact that

$$\left(\sqrt{ad} - \sqrt{bc}\right)^2 \geq 0.$$

Equality holds when $ad = bc$, and the fact that equality holds for these values shows that the number 2 cannot be replaced by a smaller number. The inequality is as "sharp" as can be.

1.13. For $a, b, c, d \geq 0$, show that $\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}$.

Solution. Since all variables are non-negative, the required result is equivalent to:

$$(a+c)(b+d) \geq ab + cd + 2\sqrt{abcd},$$

or

$$ab + cd + ad + bc \geq ab + cd + 2\sqrt{abcd},$$

or

$$ad + bc \geq 2\sqrt{abcd},$$

which follows from the fact that

$$\left(\sqrt{ad} - \sqrt{bc}\right)^2 \geq 0.$$

1.14. If a, b, c, d are all positive, and $\frac{a}{b} \leq \frac{c}{d}$, show that $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$.

Solution. Since a, b, c, d are positive, we know that $\frac{a}{b} \leq \frac{c}{d}$ implies that $ad \leq bc$, so that

$$ab + ad \leq ab + bc,$$

or

$$a(b+d) \leq b(a+c),$$

or

$$\frac{a}{b} \leq \frac{a+c}{b+d}.$$

The other half of the inequality is proved analogously. Notice that we have written the proof starting with what we know (that $\frac{a}{b} \leq \frac{c}{d}$) and we have shown that this implies the desired inequality. If you were to see our "scratchwork", you would know that we transformed both the initial and the final inequalities, then worked with the second to get the first.

But the logic must flow the other way, which is how we have written the solution.

The question of what happens when some of the variables are negative is a complicated one which can be analyzed case-by-case.

It is not hard to see that equality holds if and only if $\frac{a}{b} = \frac{c}{d}$.

This inequality forms the basis of a very interesting object called the *Farey sequence*, which readers may enjoy investigating.

1.15. Suppose we have n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$. If m and M are the smallest and largest of these fractions (respectively), show that

$$m \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq M.$$

This problem generalizes Problem 1.14.

Solution. (Watch the subscripts!) By the very definitions of m and M, we have, for every value of i,

$$m \leq \frac{a_i}{b_i} \leq M.$$

(Note that this represents n inequalities, one each for $i = 1, 2, \ldots, n$.) Hence we also have $mb_i \leq a_i \leq Mb_i$, for each i. Adding these n inequalities, we have:

$$mb_1 + mb_2 + \ldots + mb_n \leq a_1 + a_2 + \ldots + a_n \leq Mb_1 + Mb_2 + \ldots + Mb_n,$$
$$m(b_1 + b_2 + \ldots + b_n) \leq a_1 + a_2 + \ldots + a_n \leq M(b_1 + b_2 + \ldots + b_n),$$
$$m \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq M.$$

The reader who is confused by subscripts is urged to write out the inequalities for $i = 1, 2$, and 3. The reader who is used to subscripts is urged to rewrite this proof using "sigma" (Σ) notation.

1.16. Prove that for all real numbers x,

$$4(x^3 + x + 1) \leq (x^2 + 1)(x^2 + 5).$$

Solution. Multiplying out and placing everything to one side of the inequality, we have (magically!):

$$x^4 - 4x^3 + 6x^2 - 4x + 1 \geq 0.$$

The key is noting the appearance of the binomial coefficients $1, 4, 6, 4, 1$ in the polynomial. Using this hint, we have:

$$(x - 1)^4 \geq 0,$$

which is certainly true.

Note that a fourth power is the square of a square, so it must also be non-negative. This is another hint that we should compare the given polynomial with the binomial expansion noted above.

1.17. For $a, b \geq 0$, show that $a^2 - ab + b^2 \geq ab$.

Solution. We have $(a - b)^2 = a^2 - 2ab + b^2 \geq 0$, hence the result.

1.18. Make sure you know how to factor $a^3 + b^3$. Then show that for $a, b \geq 0$,

$$a^3 + b^3 \geq ab(a + b).$$

Solution. Multiply the inequality of Problem 1.17 by $a + b$.

1.19. Show that for non-negative numbers a, b, we have

$$\frac{a^3 + b^3}{2} \geq \left(\frac{a + b}{2}\right)^3.$$

Solution. Multiplying the result of Problem 1.18 by 3, and adding $a^3 + b^3$, we have

$$4a^3 + 4b^3 \geq (a+b)^3,$$

which is equivalent to the desired result.

1.20. For any non-negative number a, prove that $1 - \frac{a}{2} \leq \frac{1}{1+a^2}$.

Hint: If you run out of ideas, try factoring.

Solution. (a) Since $1 + a^2$ cannot be negative, we can multiply both sides of the inequality by it, and also by 2, to get:

$$(1+a^2)(2-a) = 2 - a + 2a^2 - a^3 \leq 2,$$

or

$$a^3 - 2a^2 + a \geq 0.$$

But this is immediate, since we have

$$a^3 - 2a^2 + a = a(a-1)^2 \geq 0.$$

1.21. Let a and b be positive real numbers. Prove that

$$\left|\frac{a^3 - b^3}{a^3 + b^3}\right| \leq 3\left|\frac{a-b}{a+b}\right|.$$

Solution. We need to show that:

$$\left|\frac{a^3 - b^3}{a^3 + b^3}\right| = \left|\frac{a-b}{a+b}\right| \cdot \left|\frac{a^2 + ab + b^2}{a^2 - ab + b^3}\right| \leq 3\left|\frac{a-b}{a+b}\right|$$

or

$$\left|\frac{a^2 + ab + b^2}{a^2 - ab + b^2}\right| \leq 3.$$

This last follows from the inequality established between the first and third expression in Problem 1.2.

Remember that a formal proof would mean reading up in this chain of computations.

1.22. Compute the product $(x+y+z)(x^2+y^2+x^2-xy-yz-zx)$.

Solution. The computation is straightforward, and the answer is amazing:

$$x^3 + y^3 + z^3 - 3xyz.$$

Most people do this by multiplying the second factor by x, then by y, then by z, and adding the results. If you proceed this way, you may notice that the three partial products are very similar. They involve the three variables in the same way, just with one replacing another. This phenomenon is called *algebraic symmetry*, and we will have much more to say about it in later chapters. For now, notice that it helps you organize and check this complicated computation.

If you didn't notice the symmetry, it might be useful to go back and redo the computation using this property of the factors.

1.23. Show that the rightmost factor in the product of Problem 1.$\boxed{22}$ can be written as $\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)$.

Solution. Again, the computation is straightforward. And again, symmetry helps organize the computation.

1.24. If x, y, z are real numbers, show that the expression

$$x^2+y^2+x^2-xy-yz-zx$$

is zero if and only if $x=y=z$.

Solution. We use the results of Problem 1.$\boxed{23}$ We can write:

$$x^2+y^2+x^2-xy-yz-zx=\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right),$$

The long factor in the right-hand side is the sum of three squares. Since a square is never negative, it can be zero if and only if each addend is zero. This means that $x=y=z$.

1.25. In a triangle whose sides have lengths a, b, c, with $a<b<c$, show that

$$a^3+b^3+c^3+3abc>2c^3.$$

Solution. From Problems 1.$\boxed{22}$ and 1.$\boxed{23}$ we know that we can write

$$z^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\left((x-y)^2+(y-z)^2+(z-x)^2\right).$$

Setting $x=a, y=b, z=-c$ in this identity, we get:

$$a^3+b^3-c^3+3abc=\frac{1}{2}(a+b-c)\left((a-b)^2+(b+c)^2+(-c-a)^2\right).$$

It is not hard to see that the right-hand side of this identity cannot be negative. Indeed, the second factor is a sum of squares, and so is non-negative. And the triangle inequality ensures us that $a+b-c>0$, so the whole product is non-negative. But this means that $a^3+b^3-c^3+3abc\geq 0$. Adding $2c^3$ to both sides gives us the required result.

1.26. (This problem requires some knowledge of complex numbers.) If we have complex numbers v, w, z such that $v^2+w^2+z^2-vw-wz-zv=0$, show that either $v=w=z$ or the points represented by v, w, z in the complex plane form an equilateral triangle.

Solution. This problem requires somewhat advanced knowledge. If you don't know about completing the square or the geometry of complex numbers, perhaps it will whet your appetite.

We multiply the given equation by 2, and use the result of Problem 1.$\boxed{23}$ to get the equivalent condition

$$(v-w)^2+(w-z)^2+(z-v)^2=0. \qquad (1.1)$$

Now (here's the trick!) we look at just the first two terms, and complete the square:

$$\begin{aligned}(v-w)^2+(w-z)^2 &= (v-w)^2+2(v-w)(w-z)\\ &\quad +(w-z)^2-2(v-w)(w-z)\\ &= [(v-w)+(w-z)]^2-2(v-w)(w-z)\\ &= (v-z)^2-2(v-w)(w-z).\end{aligned}$$

Substituting in equation (1.1), we find $2(z-v)^2-2(v-w)(w-z)=0$, or $(z-v)^2=(v-w)(w-z)$ (for three complex numbers satisfying the given condition).

Taking absolute values of these complex numbers, we find

$$|z-v|^2=|v-w||w-z|.$$

In the same way, we can show that $|v-w|^2=|w-z||z-v|$. Now we look at the three (non-negative) numbers $|v-w|, |w-z|, |z-v|$. We can show that if even one of them is zero, then they are all zero. For example, suppose $|v-w|=0$, so that $v=w$. Then

$$|v-w|^2=0=|w-z||z-v|=|v-z||z-v|=|z-v|^2,$$

so $z=v$ as well, and all three are equal. An analogous (i.e., *symmetric*) argument shows that if any other of the absolute values of the differences are 0, then they are all 0.

But if none of the absolute values of the differences are 0, we can show that they must be equal. For example, we can show that $|z-v|=|v-w|$:

$$|z-v|^2=|v-w||w-z| \quad \text{and} \quad |v-w|^2=|w-z||z-v|,$$

thus

$$\frac{|z-v|^2}{|v-w|^2}=\frac{|v-w|}{|z-v|}.$$

(By forming these fractions, we use the fact that $|z-v|$ is not zero.) Clearing fractions, we have $|z-v|^3=|v-w|^3$, or (since the cubes are real numbers) $|z-v|=|v-w|$. A similar argument holds for any other pair of these absolute values.

Now in the geometry of complex numbers, $|z-w|$ is just the length of the line segment between the points representing the complex numbers z and w. So a geometric interpretation of what we just proved is that the points representing v, w, z form an equilateral triangle.

Chapter 2

The Arithmetic-Geometric Mean Inequality, Part I

The theme of this chapter is a classic inequality called the *Arithmetic Mean-Geometric Mean Inequality* (which we will shorten to "AM-GM inequality"). This first sequence of problems concerns a version of the inequality for two variables. We will then generalize it to more than two variables.

Theorem 2.1. *The arithmetic mean of any two non-negative real numbers is greater than or equal to their geometric mean. The two means are equal if and only if the two numbers are equal.*

In other words, if $a, b \geq 0$, then

$$\frac{a+b}{2} \geq \sqrt{ab},$$

and

$$\frac{a+b}{2} = \sqrt{ab}$$

if and only if $a = b$.

Proof. The square of a real number cannot be negative. Therefore

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0.$$

But this means that

$$(\sqrt{a})^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab} \geq 0,$$

or

$$a + b \geq 2\sqrt{ab},$$

which gives us the required result. Equality holds only when $\left(\sqrt{a} - \sqrt{b}\right)^2 = 0$ which occurs when $a = b$.

Often the AM-GM inequality is used to compare a product to a sum, or to transform one into the other. Watch how this theme unfolds.

Problems

2.1. Figure 2.1 shows a semicircle with center O. Its diameter has been divided at point X into two segments of lengths $AX = a$ and $XB = b$. Which is larger, OP or XY?

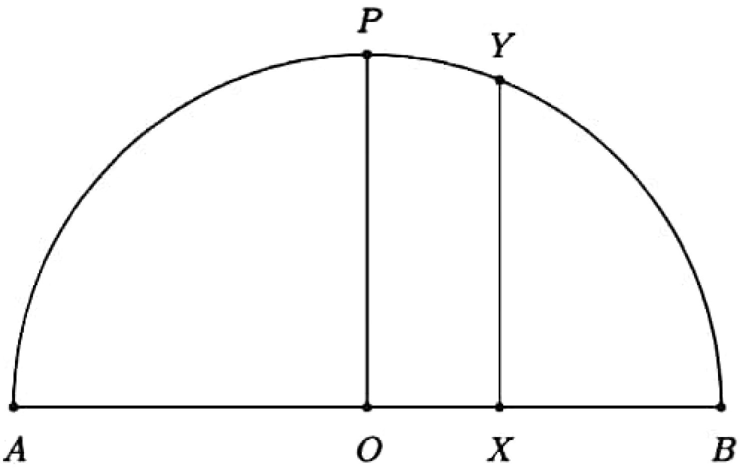

Figure 2.1

2.2. In trapezoid $ABCD$, segment MN connects the midpoints of legs AD and BC. Segment XY divides the trapezoid into two smaller trapezoids similar to each other. Figure 2.2 shows XY closer to the smaller base than to the larger base, and therefore smaller than MN. Is this correct?

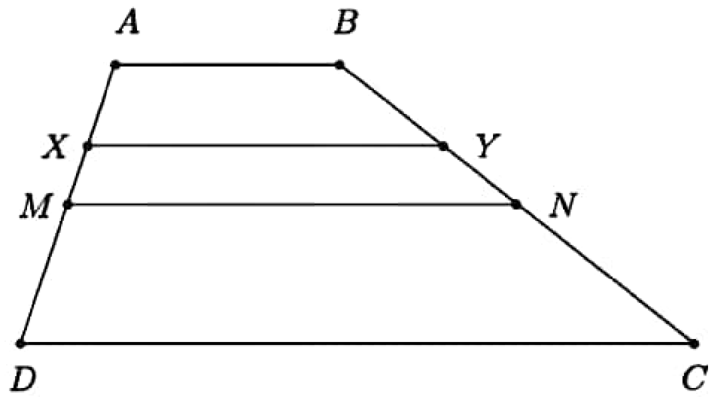

Figure 2.2

2.3. A rectangle has perimeter 20. What is its largest possible area?

2.4. A rectangle has area 100. What is its smallest possible perimeter?

2.5. Generalize the solutions to Problems 2.3 and 2.4 to show that
(a) if the sum of two positive numbers is constant, then their product is maximal when they are equal, and
(b) if the product of two positive numbers is constant, then their sum is minimal when they are equal.

2.6. During the 12 days of Christmas (in the old song), you receive not just 1 partridge in a pear tree, but 12: the gift of 1 partridge is repeated for each of 12 days. And you receive not just 2 turtle doves, but 22 turtle doves: a pair on each of the 12 days of Christmas, except the

Figure 2.2

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first. Finally, on the twelfth day, you receive 12 drummers drumming, but this gift is not repeated. Which gift do you receive the most of?

2.7. If x is a positive real number, find the smallest possible value of the expression

$$x + \frac{1}{x}.$$

2.8. If x is a positive real number, show that $2\sqrt{x} - x \le 1$.

2.9. If x is a real number, find the largest possible value of the expression

$$(x+4)(6-x).$$

2.10. If $0 \le x \le \frac{\pi}{2}$, find the smallest possible value of $\tan x + \cot x$.

2.11. For any real number x, find the largest possible value of $(\sin^2 x)(\cos^2 x)$.

2.12. If x is a real number, find the minimum value of the expression $2^x + 2^{-x}$.

2.13. If x, y, and z are non-negative real numbers, show that

$$x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \le xy + yz + xz.$$

2.14. If a, b, c, d are positive real numbers, show that

$$\sqrt{(a+c)(b+d)} \ge \sqrt{ab} + \sqrt{cd}.$$

2.15. Point D is chosen in the interior of angle $\angle ABC$. A variable line passes through D, intersecting ray BA at M and ray BC at N. Find the position of line MN that gives the smallest possible area for triangle MBN.

(For a hint to this rather difficult problem, glance at the diagram in the solution without reading the details.)

2.16. We start with n positive numbers $x_1, x_2, \ldots, x_n$, whose product is 1. Show that if we add 1 to each number, the product of the new numbers must be greater than or equal to 2^n.

2.17. For n non-negative numbers $a_1, a_2, \ldots, a_n$, show that

$$\sqrt{a_1a_2} + \sqrt{a_2a_3} + \ldots + \sqrt{a_{n-1}a_n} + \sqrt{a_na_1} \le a_1 + a_2 + \ldots + a_n.$$

2.18. Note that for $a \ge 0$, we have $\sqrt[4]{a} = \sqrt{\sqrt{a}}$. Show that if $a, b, c, d \ge 0$, then

$$\frac{a+b+c+d}{4} \ge \sqrt[4]{abcd},$$

and determine when equality occurs.

2.19. Show that if $a, b, c \ge 0$, then $\dfrac{a+b+c}{3} \ge \sqrt[3]{abc}$, and determine when equality occurs.

(Warning: You may find the proof for three variables more difficult than the corresponding proof for four variables.)

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2.17. For n non-negative numbers $a_1, a_2, \ldots, a_n$, show that

$$\sqrt{a_1a_2} + \sqrt{a_2a_3} + \ldots + \sqrt{a_{n-1}a_n} + \sqrt{a_na_1} \leq a_1 + a_2 + \ldots + a_n.$$

2.18. Note that for $a \geq 0$, we have $\sqrt[4]{a} = \sqrt{\sqrt{a}}$. Show that if $a, b, c, d \geq 0$, then

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd},$$

and determine when equality occurs.

2.19. Show that if $a, b, c \geq 0$, then $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$, and determine when equality occurs.

(Warning: You may find the proof for three variables more difficult than the corresponding proof for four variables.)

Solutions

2.1. Figure 2.1 shows a semicircle with center O. Its diameter has been divided at point X into two segments of lengths $AX = a$ and $XB = b$. Which is larger, OP or XY?

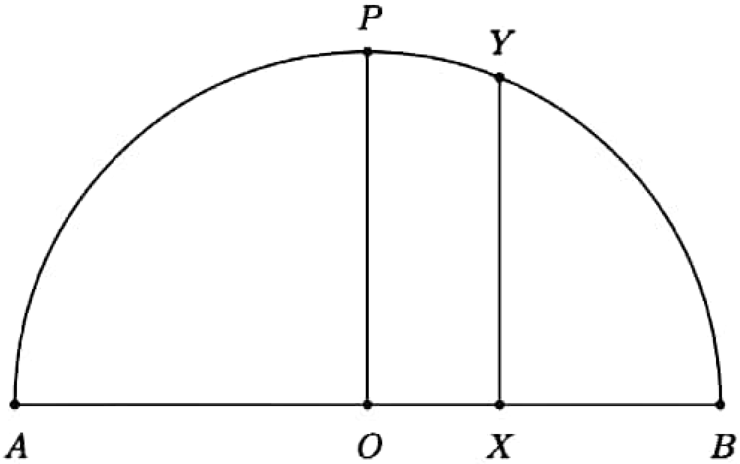

Figure 2.1

Solution. Segment XY is half of a certain chord, and OP is half of a diameter. Since a diameter of a circle is its longest chord, $OP > XY$. Analytically, this follows from (or can be considered a proof of) the main theorem, since $OP = \dfrac{a+b}{2}$ and $XY = \sqrt{ab}$.
If you're not sure why $XY = \sqrt{ab}$, note that AB is a diameter, so the angle subtended by AB at the circumference of the circle is a right angle. That is, $\angle AYB = 90°$. Also, $\angle AXY = \angle BXY = 90°$. So triangles AXY and YXB are similar. Hence

$$\frac{AX}{XY} = \frac{YX}{XB} \Rightarrow AX \cdot XB = (XY)^2 \Rightarrow XY = \sqrt{ab}.$$

2.2. In trapezoid $ABCD$, segment MN connects the midpoints of legs AD and BC. Segment XY divides the trapezoid into two smaller trapezoids similar to each other. Figure 2.2 shows XY closer to the smaller base than to the larger base, and therefore smaller than MN. Is this correct?

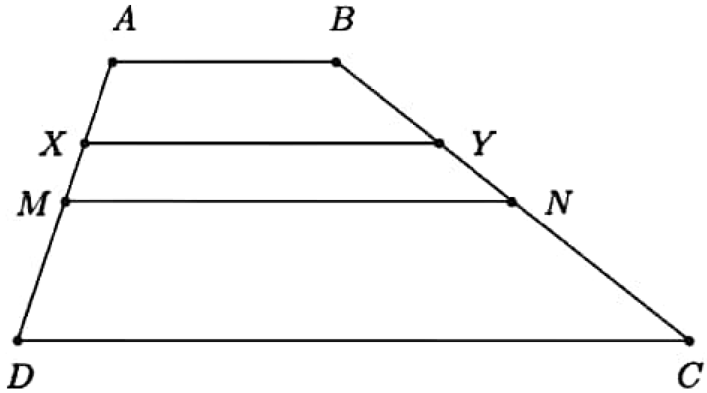

Figure 2.2

Solution. From a well-known theorem in geometry, we know that

$$MN = \frac{1}{2}(AB + CD),$$

the arithmetic mean of the bases. It is less well known that XY is the geometric mean of the bases. But it's not hard to prove.

Trapezoids $ABYX$ and $XYCD$ are similar, so their bases are in proportion: $AB : XY = XY : CD$, which means that $XY = \sqrt{AB \cdot CD}$. So our theorem tells us that $XY < MN$ and is thus closer to the smaller base.

Did you somehow think that any line parallel to the bases of a trapezoid divides into two similar trapezoids? The argument above shows that this is *not* true.

2.3. A rectangle has perimeter 20. What is its largest possible area?

Solution. If the length and width of the rectangle are denoted by a and b respectively, then we have $a + b = 10$, and we must find the maximum of ab. But the AM-GM inequality says

$$2\sqrt{ab} \le a + b = 10,$$

so that

$$ab \le \left(\frac{a+b}{2}\right)^2 = 25.$$

A quick check will show that if $a = b = 5$, then the maximum is achieved. In this case, the rectangle is a square.

2.4. A rectangle has area 100. What is its smallest possible perimeter?

Solution. If the length and width of the rectangle are denoted by a and b respectively, then we have $ab = 100$, and we must find the minimum of $2a + 2b$, or equivalently, the minimum of $a + b$. Again, the AM-GM inequality says $a + b \ge 2\sqrt{ab} = 20$, with equality if and only if $a = b = 10$. The shape of this rectangle of minimal perimeter is (again) a square, and its perimeter is 40.

2.5. Generalize the solutions to Problems 2.3 and 2.4 to show that
(a) if the sum of two positive numbers is constant, then their product is maximal when they are equal, and
(b) if the product of two positive numbers is constant, then their sum is minimal when they are equal.

Solution. The generalizations are immediate.

(a) If $a + b$ is constant, then $\left(\frac{a+b}{2}\right)^2$ is also constant, and is an upper bound for ab. The two expressions are equal if and only if $a = b$.

(b) If ab is constant, then $2\sqrt{ab}$ is also constant, and this is a lower bound for $a + b$, achieved also when $a = b$.

2.6. During the 12 days of Christmas (in the old song), you receive not just 1 partridge in a pear tree, but 12: the gift of 1 partridge is repeated for each of 12 days. And you receive not just 2 turtle doves, but 22

turtle doves: a pair on each of the 12 days of Christmas, except the first. Finally, on the twelfth day, you receive 12 drummers drumming, but this gift is not repeated. Which gift do you receive the most of?

Solution. On day n you receive n of a certain gift. And you receive that gift for a total of m days, where $m = 13 - n$. (Try it for $n = 4$, if you're not sure where the number 13 came from.) So the number of gifts of a given type that you receive is mn, where $m + n = 13$. So the maximal value for mn occurs when $m = n = 6.5$. However, m and n, from the song, must be integers. So the gift you receive the most of can be determined by finding when the values of m and n are as close to 6.5 as possible; that is, when $n = 6$ or $n = 7$. (These numbers of gifts are the same.)

Consulting the song, we find that these gifts are "geese a-laying" and "swans a-swimming".

2.7. If x is a positive real number, find the smallest possible value of the expression

$$x + \frac{1}{x}.$$

Solution. Since the product of x and $\frac{1}{x}$ is constant (it is 1), their sum is minimal when they are equal, which occurs when $x = \frac{1}{x} = 1$. Because x is positive, the smallest possible value of the expression is 2.

2.8. If x is a positive real number, show that $2\sqrt{x} - x \leq 1$.

Solution. This looks different from the previous problem. But we can make it look the same if we rewrite it so that it compares a sum (rather than a difference) to a product, which is what the AM-GM inequality does for us. Here, we can write $1 + x \geq 2\sqrt{x}$. Then, letting $a = 1$ and $b = x$ in the AM-GM inequality, we have our result.

We could also have utilized what we learned in the previous chapter to get:

$$1 - (2\sqrt{x} - x) = x + 1 - 2\sqrt{x} = (\sqrt{x} - 1)^2 \geq 0,$$

which also proves the inequality.

2.9. If x is a real number, find the largest possible value of the expression

$$(x + 4)(6 - x).$$

Solution. One could, of course, multiply this out, get a quadratic function in x, and use some standard techniques for finding the maximum. However, we can also note that $(x+4)+(6-x) = 10$, a constant, so the product of the two numbers is maximal when they are equal. This occurs when $x = 1$, and the largest possible value of the expression is 5.

Note that we have not (yet) violated the condition of the AM-GM inequality that requires both numbers to be positive. However, one might ask if we could get a still larger product if either term were negative, a

situation not covered by the AM-GM inequality. But of course, in this case, the product is negative, and our maximum value is larger. The reader is invited to explore the situation for expressions of the form $(x-a)(b-x)$ for various values of a and b.

2.10. If $0 \le x \le \frac{\pi}{2}$, find the smallest possible value of $\tan x + \cot x$.

Solution. On the domain indicated, and for any real number x for which $\tan x$ and $\cot x$ are defined, we have $(\tan x)(\cot x) = 1$. Thus their sum is minimal when $\tan x = \cot x$, which occurs when $\tan x = 1$. The required minimum value is 2.

2.11. For any real number x, find the largest possible value of $(\sin^2 x)(\cos^2 x)$.

Solution. The sum $\sin^2 x + \cos^2 x$ is constant (it equals 1). So the largest value of the given product occurs when they are equal; for example, when $x = \frac{\pi}{4}$. This largest value is $\frac{1}{4}$. Note that this implies, for $0 < x < \frac{\pi}{2}$, that the largest value of $\sin x \cos x$ is $\frac{1}{2}$. This result leads to another solution when we note that $\sin 2x = 2\sin x \cos x$, so that $\sin x \cos x = \frac{1}{2}\sin 2x$, whose maximal value is $\frac{1}{2}$.

2.12. If x is a real number, find the minimum value of the expression $2^x + 2^{-x}$.

Solution. The product $(2^x)(2^{-x})$ is constant (it is 1), so the expression is minimal when $2^x = 2^{-x}$, which occurs when $x = 0$. The minimal value is 2.

If we consider the related expression $\dfrac{e^x + e^{-x}}{2}$ (where the number e is the base of the natural logarithm), then we are studying the function $y = \cosh x$ (the hyperbolic cosine of x), which is of importance in engineering and theoretical work. Its minimal value over the real numbers is 1.

2.13. If x, y, and z are non-negative real numbers, show that

$$x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \le xy + yz + xz.$$

Solution. The square roots on the left side of the given inequality are an open invitation to apply the AM-GM inequality. We have

$$\begin{aligned} x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} &\le x\left(\frac{y+z}{2}\right) + y\left(\frac{x+z}{2}\right) + z\left(\frac{x+y}{2}\right) \\ &= xy + yz + zx. \end{aligned}$$

If only all our estimates would fall out so neatly! The two expressions are certainly equal when $x = y = z$. But are there any other possibilities for equality?

2.14. If a, b, c, d are positive real numbers, show that

$$\sqrt{(a+c)(b+d)} \ge \sqrt{ab} + \sqrt{cd}.$$

Solution. You can try writing the square roots as sums immediately, but it probably won't work. In this case, it is easier to square

Similarly, the ratio of the areas of triangles DFN and EFB is equal to the ratio of their bases, or

$$\frac{S_2}{S} = \frac{FN}{BF}.$$

Finally, since triangles MED and MBN are similar, the ratio

$$\frac{ME}{EB} = \frac{MD}{DN}.$$

Likewise, since triangles DFN and MBN are similar, we have

$$\frac{FN}{BF} = \frac{DN}{MD}.$$

(This is also true because parallel lines intercept proportional segments on any transversal.) Therefore

$$\frac{S_1}{S} \cdot \frac{S_2}{S} = \frac{ME}{EB} \cdot \frac{FN}{BF} = \frac{MD}{DN} \cdot \frac{DN}{MD} = 1.$$

Thus $S_1 S_2 = S^2$, a constant, and $S_1 + S_2$ is minimal when $S_1 = S_2 = S$. This happens when MN is parallel to EF.

2.16. We start with n positive numbers $x_1, x_2, \ldots, x_n$, whose product is 1. Show that if we add 1 to each number, the product of the new numbers must be greater than or equal to 2^n.

Solution. We know $x_1, x_2, \ldots, x_n = 1$, and we want to show that

$$(1 + x_1)(1 + x_2) \ldots (1 + x_n) \geq 2^n.$$

We can use the AM-GM inequality to transform each sum on the left to a product. We have

$$1 + x_1 \geq 2\sqrt{1 \cdot x_1} = 2\sqrt{x_1},$$

and similarly for the other factors. Multiplying, we find that

$$(1 + x_1)(1 + x_2) \ldots (1 + x_n) \geq (2\sqrt{x_1})(2\sqrt{x_2}) \ldots (2\sqrt{x_n}) = 2^n \cdot 1,$$

which is the result we need.

2.17. For n non-negative numbers $a_1, a_2, \ldots, a_n$, show that

$$\sqrt{a_1 a_2} + \sqrt{a_2 a_3} + \ldots + \sqrt{a_{n-1} a_n} + \sqrt{a_n a_1} \leq a_1 + a_2 + \ldots + a_n.$$

Solution. We have

$$\begin{aligned} \sqrt{a_1 a_2} &\leq \frac{a_1 + a_2}{2}, \\ \sqrt{a_2 a_3} &\leq \frac{a_2 + a_3}{2}, \\ &\vdots \\ \sqrt{a_{n-1} a_n} &\leq \frac{a_{n-1} + a_n}{2}, \\ \sqrt{a_n a_1} &\leq \frac{a_n + a_1}{2}. \end{aligned}$$

Adding, and noting that there are two copies of each addend a_i in the right-hand column, we have the required result.

2.18. Note that for $a \geq 0$, we have $\sqrt[4]{a} = \sqrt{\sqrt{a}}$. Show that if $a, b, c, d \geq 0$, then

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd},$$

and determine when equality occurs.

Solution. We have

$$\begin{aligned}\frac{a+b+c+d}{4} &= \frac{(a+b)+(c+d)}{4}\\ &= \frac{1}{2}\cdot\frac{a+b}{2}+\frac{1}{2}\cdot\frac{c+d}{2}\\ &\geq \frac{1}{2}\sqrt{ab}+\frac{1}{2}\sqrt{cd}\\ &= \frac{\sqrt{ab}+\sqrt{cd}}{2}\\ &\geq \sqrt{\sqrt{abcd}}\\ &= \sqrt[4]{abcd}.\end{aligned}$$

Equality occurs when $a = b = c = d$.

2.19. Show that if $a, b, c \geq 0$, then $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$, and determine when equality occurs.

(Warning: You may find the proof for three variables more difficult than the corresponding proof for four variables.)

Solution I. We can apply the result of 2.18 by reducing four variables to three. When is the average (arithmetic mean) of four variables equal to the arithmetic mean of three of them? When the missing (fourth) variable is equal to the average of the other three! So we apply the result of 2.18 to the four numbers a, b, c, $\dfrac{a+b+c}{3}$. We have, as expected:

$$\begin{aligned}\frac{a+b+c+\frac{a+b+c}{3}}{4} &= \frac{\frac{3a}{3}+\frac{3b}{3}+\frac{3c}{3}+\frac{a+b+c}{3}}{4}\\ &= \frac{4a+4b+4c}{12} = \frac{a+b+c}{3}.\end{aligned}$$

And we know this is greater than or equal to $\sqrt[4]{abc\left(\dfrac{a+b+c}{3}\right)}$.

We must now reduce the fourth root to a cube root. We can do this by recognizing that the expression $\dfrac{a+b+c}{3}$ is on both sides of our

inequality. That is, letting $\beta = \dfrac{a+b+c}{3}$, we have

$$\beta \geq \sqrt[4]{abc\beta}, \quad \text{or} \quad \beta^4 \geq abc\beta, \quad \text{or} \quad \beta^3 \geq abc,$$

which is equivalent to what we wanted to prove. Whew!

Solution II. This solution is nicer, but more difficult to think of. We use the results from Problems 1.22 and 1.23.

From those problems, we know that

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)\left((x-y)^2 + (y-z)^2 + (z-x)^2\right).$$

If $x, y, z \geq 0$, then the right-hand side of this identity is certainly non-negative. But this means that $x^3 + y^3 + z^3 - 3xyz \geq 0$.

Now we can substitute $x = \sqrt[3]{a}, y = \sqrt[3]{b}, z = \sqrt[3]{c}$, to get

$$a + b + c - 3\sqrt[3]{abc} \geq 0,$$

which gives us the inequality we want. Equality holds when the three squares $(x-y)^2, (y-z)^2, (z-x)^2$ are all zero, which implies that $a = b = c$.

But without Problems 1.22, 1.23, how would you have thought of this?

Chapter 3

The Arithmetic-Geometric Mean Inequality, Part II

We begin this chapter with an interlude: a story-proof of the AM-GM inequality for any set of non-negative numbers. We then look at problems using the AM-GM inequality for three variables. Many of these generalize to any number of variables in ways that offer no difficulty, once the reader has worked the case $n = 3$. Finally, we offer a few advanced problems using the AM-GM inequality.

Interlude: Cauchy's Great-Granddaughter

We have looked at the AM-GM inequality in two variables: that is, if $a, b \geq 0$, then $\sqrt{ab} \leq \dfrac{a+b}{2}$. We have also looked at the corresponding inequalities for three numbers, and four numbers:

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3},$$

$$\sqrt[4]{abcd} \leq \frac{a+b+c+d}{4}.$$

It is in fact true generally that for any n:

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

An early proof of this statement is credited to Augustin-Louis Cauchy, and is one of the most beautiful in the literature. But we search in vain for Cauchy's motivation in doing this. In the best mathematical tradition, Cauchy was given to hiding his scratch work and showing only his results.

And, in fact, while some mathematicians (Poincaré, Pólya) did allow us into their mental workshop, Cauchy did not. Cauchy was not exactly a people person. He was deeply conservative in his habits and beliefs. (He followed the French King Charles X into exile after the revolution of 1830.) So even if we could conjure him up, it is not likely he would reveal his mind to us.

Instead, let us conjure up his great-great-great-granddaughter, Augusta-Louise Cauchy[1].

Her cumbersome first name is an issue. She never liked it. Her parents call her Gussie-Lou, but she bridles at that. Her grandparents wanted "Moon Unit", not realizing how it dated them. And "Augie" sounds like a cartoon character. We'll settle for "Augusta-Louise".

Augusta-Louise is a junior in an American high school. But we should let her talk:

"It's so embarrassing. My math teachers always tell everyone about my last name. And they pronounce it wrong. It's co-SHEE, not CO-shee. The French teacher, Mme. de Trop, gets it right, but I don't take French. Japanese.

"And it's a good thing I'm OK in math, or I'd never hear the end of it. 'CO-shee this. CO-shee that. Why don't you live up to your name?' Thank goodness I'm spared those silly comments.

"So anyway, I have this extra-credit problem, to prove the AM-GM inequality. And the teacher said that CO-shee already proved it. OK. So I have to live up to my

"Can you use the induction hypothesis twice in the same proof? I guess you can. If we assume it's true, it can't suddenly become false. So I can write:

$$\begin{aligned}
\sqrt[4]{a_1a_2a_3a_4} &= \sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}} \\
&\le \frac{\sqrt{a_1a_2}+\sqrt{a_3a_4}}{2} \\
&\le \frac{\dfrac{a_1+a_2}{2}+\dfrac{a_1+a_2}{2}}{2} \\
&= \frac{a_1+a_2+a_3+a_4}{4}.
\end{aligned}$$

"And it's neat how the algebra works.

"The same thing will get us from $n = 4$ to $n = 6$. NO! It will get us from $n = 4$ to $n = 8$. I don't know how to get $n = 6$. But it will also get us from $n = 8$ to $n = 16$. It really is just an easy induction. The algebra is the same. I'll need subscripts, which I hate. But OK. At least I proved it for powers of 2, for $n = 2^k$."

"Can you use the induction hypothesis twice in the same proof? I guess you can. If we assume it's true, it can't suddenly become false. So I can write:

$$\begin{aligned}
\sqrt[4]{a_1a_2a_3a_4} &= \sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}} \\
&\le \frac{\sqrt{a_1a_2}+\sqrt{a_3a_4}}{2} \\
&\le \frac{\dfrac{a_1+a_2}{2}+\dfrac{a_1+a_2}{2}}{2} \\
&= \frac{a_1+a_2+a_3+a_4}{4}.
\end{aligned}$$

"And it's neat how the algebra works.

"The same thing will get us from $n = 4$ to $n = 6$. NO! It will get us from $n = 4$ to $n = 8$. I don't know how to get $n = 6$. But it will also get us from $n = 8$ to $n = 16$. It really is just an easy induction. The algebra is the same. I'll need subscripts, which I hate. But OK. At least I proved it for powers of 2, for $n = 2^k$."

So Augusta-Louise (we can call her that, because she's not listening) wrote it all out, with subscripts. She had to be careful, but the same algebra, and the same logic, that brought her from 2 to 4 brought her from 2^k to 2^{k+1}. She showed her math teacher, who praised her. But Augusta-Louise wondered out loud how to get $n = 3$. The math teacher wouldn't tell her. "Go look in a textbook", he said.

So she did. And she saw the phrase 'backwards induction'.

"How can induction work backwards?" Augusta-Louise wondered. "Well, it can. Sort of. You can prove that if it's true for $n+1$ then it's true for n. Or, you could say the same thing by proving, 'If it's true for n, then it's true for $n-1$.' What good does that do?"

Augusta-Louise sat and thought about it for a while.

Then: "OMG! That would work! If I can prove that n implies $n-1$, then from 16 we can get to 15, then to 14, and so on. I mean, let's say it's right. From $n = 2^k$ we can get to $n = 2^k - 1$, then to $n = 2^k - 2$, and all the way down to 2^{k-1}. Then it also proves it from 2^{k-1} down to 2^{k-2}, and so on, all the way down to $n = 3$ and $n = 5$ and $n = 6$, which I was missing. That. Is. So. Cool.

"OK. Let's do it. I'll practice going from 6 to 5, then write it in general. We have to prove that:

$$\text{if } \sqrt[6]{a_1a_2a_3a_4a_5a_6} \le \frac{a_1+a_2+a_3+a_4+a_5+a_6}{6},$$

for any values of $a_1, a_2, \ldots, a_6$,

$$\text{then } \sqrt[5]{b_1b_2b_3b_4b_5} \le \frac{b_1+b_2+b_3+b_4+b_5}{5}.$$

"OK. This is gonna be hard. How do we take the sixth root of six things, and make them into the fifth root of five things? It's gonna be hard.

"Wait. Wait. Last time I made the left sides the same. Maybe I should make the right sides the same this time. Let's see. The right side is just an average, like the average of five or six tests."

That analogy came easily to Augusta-Louise, as it does to most students.

"I know what to do. I can do what Mme. de Trop does. I helped her once. She has this computer program for averaging six tests. But if a student is absent, and took only five tests, she doesn't give a makeup. She asked me, 'What's a fair grade to enter for the sixth test?' And that was easy. You just enter the average of the five tests as the score for the sixth test. Everyone knows that."

Well, most people (except math teachers) don't know that, but usually understand it once a student like Augusta-Louise points it out to them.

"So: I can let $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$, $a_5 = b_5$ and try

$$a_6 = \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5}.$$

With any luck, the algebra will work out."

(Of course it works out, or you wouldn't be reading this.)

"Let's see. I'm going to start writing the inequalities the 'other way', with the arithmetic mean on the left and a '$\geq$' sign in between. I find it's easier for me to think that way, and there's no real difference. We have:

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} = \frac{b_1 + b_2 + b_3 + b_4 + b_5 + \dfrac{b_1 + b_2 + b_3 + b_4 + b_5}{5}}{6}.$$

Daunting. But let's plug on. In terms of the b's, the left-hand side is:

$$\begin{aligned}
&\frac{b_1 + b_2 + b_3 + b_4 + b_5 + \dfrac{b_1 + b_2 + b_3 + b_4 + b_5}{5}}{6} \\
&= \frac{5b_1 + 5b_2 + 5b_3 + 5b_4 + 5b_5 + b_1 + b_2 + b_3 + b_4 + b_5}{30} \\
&= \frac{6b_1 + 6b_2 + 6b_3 + 6b_4 + 6b_5}{30} = \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5},
\end{aligned}$$

which is just what I told Mme. de Trop.

"OK. Now for the right-hand side. It doesn't look great. We know, from the a's, that

$$\begin{aligned}
\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} &= \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5} \\
&\geq \sqrt[6]{a_1 a_2 a_3 a_4 a_5 a_6} \\
&= \sqrt[6]{b_1 b_2 b_3 b_4 b_5 \left(\frac{b_1 + b_2 + b_3 + b_4 + b_5}{5}\right)}.
\end{aligned}$$

"But what do we do now? How do we turn a sixth root into a fifth root?

"Wait. The same thing appears on the left and the right. We can divide by it, in a funny way. Let's let

$$\beta = \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5},$$

the arithmetic mean of the b's. Then all we really have is:

$$\beta \geq \sqrt[6]{b_1 b_2 b_3 b_4 b_5 \beta},$$

or

$$\beta^6 \geq b_1 b_2 b_3 b_4 b_5 \beta,$$

or

$$\beta^5 \geq b_1 b_2 b_3 b_4 b_5,$$

and

$$\beta \geq \sqrt[5]{b_1 b_2 b_3 b_4 b_5}.$$

"And that's what I need to prove. Yesss! Extra credit!"

Augusta-Louise smiled proudly at herself for a minute. She wrote it up in general, for any n. Then a look came across her face. "Why did that last step work? Was I just lucky?"

Maybe a reader can give a satisfactory answer to that. It's the one step in the proof that Augusta-Louise can't tell us about, any more than her great-great-great-grandfather can.

Problems

3.1. For positive numbers a, b, c, x, y, z show that

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}.$$

3.2. Show that if a, b, c are positive numbers such that $a + b + c = 1$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

3.3. More generally, for positive numbers a, b, c, show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}.$$

3.4. If $a_1, a_2, \ldots, a_n$ are n positive numbers, show that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2.$$

3.5. If $a_1, a_2, \ldots, a_n$ are positive numbers, show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} \geq n.$$

3.6. Let a, b, c, d be positive real numbers such that $a+b+c+d=1$. Prove that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\left(1+\frac{1}{d}\right) \geq 625.$$

Can you extend this result to n variables?

Solutions

3.1. For positive numbers a, b, c, x, y, z show that

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}.$$

Solution. We can cube both sides, to get the equivalent inequality

$$(a+x)(b+y)(c+z) \geq \left(\sqrt[3]{abc} + \sqrt[3]{xyz}\right)^3.$$

Then we transform the two sides of the inequality separately. We have:

$$\begin{aligned}(a+x)(b+y)(c+z) &= (a+x)(bc+bz+yc+yz)\\ &= abc+abz+acy+ayz+bcx+bxz+cxy+xyz\\ &= abc+xyz+abz+acy+ayz+bcx+bxz+cxy.\end{aligned}$$

On the right-hand side we have

$$\left(\sqrt[3]{abc} + \sqrt[3]{xyz}\right)^3 = abc+xyz+3\left(\sqrt[3]{a^2b^2c^2xyz} + \sqrt[3]{abcx^2y^2z^2}\right).$$

Comparing the two results, we see that the inequality

$$abz+acy+ayz+bcx+bxz+cxy \geq 3\left(\sqrt[3]{a^2b^2c^2xyz} + \sqrt[3]{abcx^2y^2z^2}\right)$$

is again equivalent to the one we want to prove. This (finally!) resembles a previous result, that of Problem 2.19. We can separate the six terms on the left into two sets of three so that their products are just what is underneath each radical, and from Problem 2.19, we have:

$$\frac{abz+acy+bcx}{3} \geq \sqrt[3]{a^2b^2c^2xyz},$$

$$\frac{ayz+bxz+cxy}{3} \geq \sqrt[3]{abcx^2y^2z^2}.$$

Adding these two inequalities, and following the argument backwards, we obtain the required result.

Notes:

(a) Can you really cube both sides of an inequality to get an equivalent inequality? Yes, because $y=x^3$ is a continuous function which is always increasing. In any such situation, you can take the functional value of both sides of inequality to get an equivalent inequality.

(b) Do you see why this principle does not allow us always to square both sides of an inequality? We don't always get an equivalent inequality, because the argument will not run backwards.

(c) Go back and look at the argument to see that the six variables concerned all entered into the original inequality symmetrically, and that we treated them symmetrically at each step.

3.2. Show that if a, b, c are positive numbers such that $a + b + c = 1$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

Solution. Using the AM-GM inequality for three variables, we can write:

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$

Since $a + b + c = 1$, this can be written as

$$\frac{1}{\sqrt[3]{abc}} \geq 3.$$

Using this inequality, and applying the AM-GM inequality a second time, we have

$$\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \geq \frac{1}{\sqrt[3]{abc}} \geq 3.$$

The result follows.

3.3. More generally, for positive numbers a, b, c, show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}.$$

Solution. From the AM-GM inequality, we have:

$$a + b + c \geq 3\sqrt[3]{abc}$$

and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c}} = \frac{3}{\sqrt[3]{abc}}.$$

Multiplying these two inequalities, we get the desired result. Equality holds occurring only when $a = b = c$.

Notes: This is a tricky problem. The hint is that it is placed here, right after the AM-GM inequality for three variables. The sum of three addends gives us another clue. The solution is essentially motivated by comparing each side to $\sqrt[3]{abc}$.

Were the variables used symmetrically in this (very tricky) solution?

3.4. If $a_1, a_2, \ldots, a_n$ are n positive numbers, show that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2.$$

Solution. From the general AM-GM inequality, we have:

$$a_1 + a_2 + \ldots + a_n \geq n\sqrt[n]{a_1 a_2 \ldots a_n}$$

and

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\geq\frac{n}{\sqrt[n]{a_1a_2\ldots a_n}}.$$

We get the required result by multiplying these two inequalities, with equality just when $a_1=a_2=\ldots=a_n$.

3.5. If $a_1,a_2,\ldots,a_n$ are positive numbers, show that

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1}\geq n.$$

Solution. By the generalized AM-GM inequality, we have:

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1}\geq n\sqrt[n]{\frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\cdot\frac{a_3}{a_4}\ldots\frac{a_n}{a_1}}=n\sqrt[n]{1}=n.$$

The case for equality is not the usual. The two expressions are equal when

$$\frac{a_1}{a_2}=\frac{a_2}{a_3}=\ldots=\frac{a_n}{a_1}=1.$$

3.6. Let a,b,c,d be positive real numbers such that $a+b+c+d=1$. Prove that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\left(1+\frac{1}{d}\right)\geq 625.$$

Can you extend this result to n variables?

Solution. Since this is a chapter on the AM-GM inequality, our first instinct might be to write $1+\frac{1}{a}\geq 2\sqrt{\frac{1}{a}}$, and rewrite the other terms similarly. This turns out not to work too well, as we might suspect. First of all, this set of problems is about the AM-GM inequality for more than two variables. But more mathematically, equality holds when $a=1$, which leaves no 'room' for the other variables: The four of them sum to 1.

But looking at the case for equality will give us a hint. Why does the inequality we desire have the number 625? Well, this is 5^4, and if all the factors on the left were equal, they would all have to be equal to 5. And in fact if $a=b=c=d=\frac{1}{4}$, the inequality does turn into an equality.

We can continue to think of the case for equality, as splitting $1+\frac{1}{a}$ into five pieces, all of which are equal if $a=b=c=d$. Since $a+b+c+d=1$, we can write:

$$1+\frac{1}{a}=1+\frac{a+b+c+d}{a}=1+1+\frac{b}{a}+\frac{c}{a}+\frac{d}{a}\geq 5\sqrt[5]{\frac{bcd}{a^3}},$$

by the AM-GM inequality for several variables. Similarly,

$$1+\frac{1}{b}\geq 5\sqrt[5]{\frac{cda}{b^3}},$$

$$1+\frac{1}{c} \geq 5\sqrt[5]{\frac{dab}{c^3}},$$

$$1+\frac{1}{d} \geq 5\sqrt[5]{\frac{abc}{d^3}}.$$

Multiplying these four inequalities yields the conclusion.

And the result can easily be extended: If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$, then

$$\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\ldots\left(1+\frac{1}{a_n}\right) \geq (n+1)^n.$$

Chapter 4

The Harmonic Mean

Introduction

Here are a few typical "word problems" from algebra textbooks.

Example 4.1. Dora walked to school, a distance of 1 mile, at a rate of 5 miles per hour. She got a lift home in a car which went 15 miles per hour. What was her average rate for the round trip?

Solution. Most people, without thinking much, would say that the average rate is $\frac{5+15}{2} = 10$ MPH. What is wrong with this?

Well, let us look at how much time she took going and coming. Going to school she took $\frac{1}{5}$ hour $= 12$ minutes walking. (In algebra, $T = \frac{D}{R} = \frac{1}{5}$ hours, but it's easier than that if you think arithmetically.) Coming home she took $\frac{1}{15}$ hour $= 4$ minutes. So altogether she took $12 + 4 = 16$ minutes $= \frac{4}{15}$ hours.

A reasonable way to think of the "average rate" is to say that if she somehow managed to go and come at this same average rate r (say, by bicycle), then the round trip at rate r should take her the same time as the trip coming and going at two different rates. (Stop and think if you agree with this idea of what "average" means.)

Time is distance divided by rate. So if she travels at this average rate r, she will be going a distance of 2 miles, and $\frac{2}{r} = \frac{4}{15}$, so $r = \frac{15}{2}$ or 7.5 MPH. (You can check that this works: if she goes 1 mile to school and 1 mile back at this rate, she will spend 16 minutes $= \frac{4}{15}$ hours traveling.)

Note that this is less than the "naïve average" of 10 MPH. This makes sense: Dora is spending much more time walking than riding, and somehow the "average" we take should reflect this.

The point is that when we talk about "averaging" rates, we are not talking about the same operation as "averaging" test grades. When we average two test grades, we are asking what single grade on both tests would measure the same achievement as two different grades. This kind of average is the arithmetic mean of the two grades.

When we average two rates (over the same distance) we are asking which single rate would make up, in time, for the two different rates. This kind of average is called the *harmonic mean* of the two rates. (The term "average" is not really a mathematical term. It means any concept of a "typical" or "usual" value. How we actually compute this "typical" value depends on how we use the value.)

Let us look at this harmonic mean in general. Suppose we have two trips of the same distance d miles (in the example above, the fact that we had a round trip guaranteed that the two distances were equal). Suppose one trip is made at the rate f (for fast) MPH and another at the rate s (for slow) MPH. Let us compute the time taken in various legs of the journey.

For the fast leg, the time is $\frac{d}{f}$. For the slow leg, the time is $\frac{d}{s}$. And if the full journey (a distance $2d$) is taken at an average rate r, the time is $\frac{2d}{r}$. So we must have:

$$\frac{d}{f} + \frac{d}{s} = \frac{2d}{r}.$$

But look! The d's cancel out. (The average rate should be the same whether we go 1 mile or 100 miles.) And we get:

$$\frac{2}{r} = \frac{1}{f} + \frac{1}{s}.$$

This is the arithmetic definition of the harmonic mean of two numbers. We can rewrite it in several ways. One way is:

$$\frac{1}{r} = \left(\frac{1}{2}\right)\left(\frac{1}{f} + \frac{1}{s}\right).$$

That is, the reciprocals (multiplicative inverses) of the rates are averaged in the usual way: The reciprocal of r is half the sum of the reciprocals of f and s. This make algebraic sense, because when we compute the time of a particular trip, the rates appear in the denominator, not in the numerator. So we sort of have to "average the denominators", which requires this unusual arithmetic.

Another way to write the harmonic mean is:

$$r = \frac{2}{\frac{1}{f} + \frac{1}{s}}. \tag{4.1}$$

Yet another way to write this expression, which is not nearly so informative, is:

$$r = \frac{2fs}{f + s}.$$

This is sometimes given as the formal definition of the harmonic mean.

You can memorize the formula for the harmonic mean, and make average rate problems into plug-ins. But you will probably get more confused than if you think them through more logically.

It's more than just $D = RT$

The harmonic mean shows up in solving many types of verbal problems, and not just in $D = RT$ problems. For example, let us look at a problem about filling a swimming pool.

Example 4.2. One pipe can fill a swimming pool in 2 hours. Another can fill it in 3 hours. If both pipes are turned on, how long will it take to fill the pool?

Solution. Let us again look at a "naïve" solution. Well, the easiest, and most wrong solution, is to say that the first pipe takes 2 hours and the second pipe takes 3 hours, so together it will take them 5 hours. This is silly. How can it take them longer working together than either took alone? A better — but still naïve and wrong — solution is to say that if they work together they do 5 hours "worth" of work, so each one has to do $5/2 = 2.5$ hours of work, and it will take them that long to fill the tank. Note that this is the "testing average" (the mathematical term is the *arithmetic mean*) of the two rates.

But in fact the faster pipe does more work. Thus the two pipes shouldn't be counted equally in the "average". What is going on? We can see this a little more clearly if we think of this as a problem about rates. The first pipe fills the pool in two hours, so in one hour it fills $\frac{1}{2}$ the pool. That is, it fills pools at a rate of $\frac{1}{2}$ pool per hour. The second pipe fills the pool in three hours, so it fills pools at the rate of $\frac{1}{3}$ pool per hour. We want to find an average rate at which a pipe could fill the pool which matches these two rates.

Now we can add: together, the two pipes fill $\frac{1}{2}+\frac{1}{3}$ of the pool, or $\frac{5}{6}$ of the pool, in one hour. What rate does this correspond to? That's asking how many times does $\frac{5}{6}$ fit into 1 (pool), and that's just division. The answer is $\frac{1}{\frac{5}{6}}$ or $\frac{6}{5}$ of an hour, and this is the time it takes for both pipes to fill the pool together.

Note that this is half the harmonic mean of 2 and 3.

We can see what is going on if we compare this problem to $D = RT$ problems. In this one, we can think of the first pipe as filling the pool, from the bottom, at a rate of $\frac{1}{2}$ pool per hour. If you like, think of the water level traveling at this rate up the side of the pool.

Now for a little imagination. We can think of the second pipe as "filling" the pool from the top down. So the water coming out of this pipe magically floats at the top of the pool, and gets deeper as the pool fills. Using this model, the water level is traveling downwards, at a rate of $\frac{1}{3}$ pool per hour (since it would take 3 hours to travel all the way down).

So we have a $D = RT$ problem in which the two travelers are going towards each other. Let's make this more specific.

Example 4.3. Tom and Jerry live a mile apart. They start walking towards each other's house, at constant rates, along the same path. In t hours, Tom walks 1 mile, and in j hours, Jerry walks 1 mile. How long will it be before they meet?

Solution. As you might suspect, it's the same problem. Tom walks $\frac{1}{t}$ miles, that is, $\frac{1}{t}$ of the path, in one hour, while Jerry walks $\frac{1}{j}$ miles in one hour. So together, in one hour, they walk $\frac{1}{t} + \frac{1}{j} = \frac{t+j}{tj}$ miles of the path, or at that rate in miles per hour. How many hours does it take them to walk the path? That's just asking how many times $\frac{t+j}{tj}$ goes into 1 mile, which is $\frac{1}{\frac{t+j}{tj}} = \frac{tj}{t+j}$ hours.

Sound familiar? The algebra is just the same as when we were filling the pool. The answer is half the harmonic mean of the two rates. Half, because they are not traversing the path twice. Similarly, in filling the pool, the result is half the harmonic mean of the two rates because we are not filling then emptying the pool.

Example 4.4. Suppose you spend \$6 on pink pills costing a cents per dozen, and \$6 on blue pills costing b cents per dozen. What was the average price per dozen of the pills you've bought?

Solution. Sigh. It's going to be the harmonic mean, of course. But let's see why.

The average price will be the total amount spent divided by the total number of pills we bought. We spent \$12 altogether, but on how many pills? Well, we bought the pink pills in dozens, each one costing a cents. How many dozens did we buy? That's asking how many times a goes into 600: $\frac{600}{a}$. Similarly, the number of dozens of blue pills is $\frac{600}{b}$, and the total number of pills is the sum of these, or $\frac{600a + 600b}{ab}$. We must divide this into \$12 $= 1200$ cents to get our average, which is $\frac{1200ab}{600a + 600b} = \frac{2ab}{a+b}$ cents per dozen, the harmonic mean of a and b.

The reader is invited to check that if each dozen pills had cost this much, we would have spent the same amount of money.

Notes and Summary

For two positive numbers a and b, we call $\frac{2ab}{a+b}$ the *harmonic mean* of a and b.

Note that if h is the harmonic mean of a and b, we have $\frac{2}{h} = \frac{1}{a} + \frac{1}{b}$.

Note further that if $\frac{1}{k} = \frac{1}{a} + \frac{1}{b}$, then k is *half* the harmonic mean of a and b.

Problems

4.1. Show that the geometric mean of a and b is also the geometric mean of $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$.

4.2. Show that if AM, GM, HM are respectively the arithmetic, geometric, and harmonic means of two positive numbers, then $HM \le GM \le AM$.

4.3. We have so far begged an important question. Is the harmonic mean of a and b always a mean? That is, is it always between a and b? Show that if $0 \le a < b$, then $a < \frac{2ab}{a+b} < b$.

4.4. **a.** Look back at Figure 2.1, and connect O to Y as shown in Figure 4.1 below. Let point Z be the foot of the perpendicular from X to OY. Using the same segment lengths as given in Problem 2.1, show that YZ is the harmonic mean of a and b. Deduce geometrically that the harmonic mean of a and b is less than or equal to their arithmetic mean.

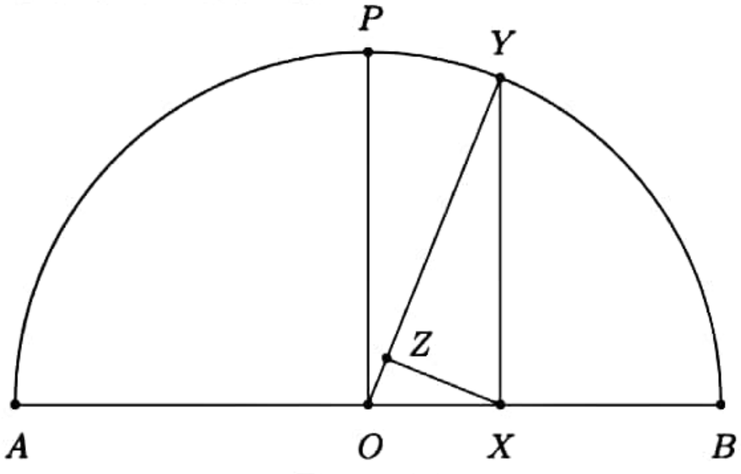

Figure 4.1

b. (This problem requires the 'triangle inequality', a geometric inequality which we haven't discussed in this volume.) Let h_a, h_b, h_c be the lengths of the altitudes of a triangle (with sides a, b, c). Prove that $2h_a$ is greater than the harmonic mean of h_b and h_c.

The Harmonic Mean of Several Quantities

Just as we can take the arithmetic or geometric mean of several quantities, we can also take the harmonic mean of several quantities. Recall that

we can define the harmonic mean h of a and b by the equation

$$h = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

(i.e., equation (4.1)). Likewise, the harmonic mean h of a, b, and c is defined by:

$$h = \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

And in general, for n positive numbers $a_1, a_2, \ldots, a_n$, the harmonic mean is defined by:

$$h = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}.$$

And now we have some work.

Problems

4.5. Suppose we have two sets of n positive numbers each:

$$S_1 = \{a_1, a_2, a_3, \ldots, a_n\} \text{ and } S_2 = \{a_1, x, a_3, \ldots, a_n\}.$$

If $a_2 > x$, show that the harmonic mean of the numbers in S_1 is greater than the harmonic mean of those in S_2.

That is, replacing a number in a set by a smaller number decreases the harmonic mean of the set.

4.6. Show that the harmonic mean of n numbers is between the largest and the smallest of them.

4.7. If AM, GM, HM stand respectively for the arithmetic, geometric, and harmonic means of n positive numbers, show that $AM \geq GM \geq HM$. (We have already shown the first inequality in Chapter 3. Show the second.)

The Harmonic Mean in Geometry

For this next set of problems, the term "find" means "estimate the location of". Unless otherwise indicated, it does not mean "construct by straightedge and compass" (or other tools).

Problems

4.8. Given two points A and B on a line, find a point W on line segment AB such that $AW = WB$.

4.9. Given two points A and B on a line, find a point X on line segment AB such that $BX = 2XA$.

4.10. Given two points A and B on a line, find a point Y on line segment AB such that $AY = 2YB$.

4.11. Given two points A and B on a line, find another point Y' on line AB (distinct from the point Y in Problem 4.10) such that $AY' = 2Y'B$.

4.12. Are there any other points Z, anywhere in the plane, other than Y' (from Problem 4.11) such that $AZ = 2BZ$?

4.13. (Extra credit) Given two points A and B on a line, construct, with straightedge and compass, the points Y and Y' referred to in Problems 4.10 and 4.11.

♦

In Problem 4.10 we started with a point Y between A and B such that $AY = 2YB$. We then found another point Y' on line AB (but outside segment AB) such that $AY' = 2Y'B$. We say that Y and Y' are *harmonic conjugates* with respect to segment AB. In general, if we have a line segment AB, and a given ratio r, there are two points Y, Y' on line AB such that $AY : YB = AY' : Y'B = r$. The term 'harmonic conjugates' is used for the pair Y, Y', no matter what the ratio r may be.

♦

4.14. If X and Y are harmonic conjugates with respect to A and B, show that A and B are also harmonic conjugates with respect to X and Y.

4.15. Generalize Problem 4.13. Given points A and B, and a point X between them, construct, with straightedge and compass, the point Y that is the harmonic conjugate of X. What if point X is given outside segment AB?

4.16. Let X and Y be harmonic conjugates with respect to A and B. Let $AX = a$, $AB = h$, and $AY = b$. Show that h is the harmonic mean of a and b.

4.17. For two points A and B, where is the harmonic conjugate of the midpoint of AB?

4.18. If you ride your bike up a hill at 15 MPH, at what rate must you ride down the hill, so that your average speed for the two trips is 30 MPH? If you have trouble with this problem and Problem 4.17, you are in good company.
Albert Einstein is on record for finding the solution counterintuitive. But can you make a geometric diagram which shows what is happening?

Solutions

4.1. Show that the geometric mean of a and b is also the geometric mean of $\dfrac{a+b}{2}$ and $\dfrac{2ab}{a+b}$.

Solution. It's just algebra, and it's easy if we put it in the following form:

$$\left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab,$$

or that the arithmetic mean times the harmonic mean is the square of the geometric mean. The truth of this equation is easy to see from the form of the fractions.

4.2. Show that if AM, GM, HM are respectively the arithmetic, geometric, and harmonic means of two positive numbers, then $HM \leq GM \leq AM$.

Solution. This is again immediate: we know that the geometric mean of two numbers is between the larger and smaller of them, and this is all that the inequality says. The means are equal, of course, when the numbers are equal.

We have shown that the geometric mean of two positive real numbers lies between their harmonic mean and their arithmetic mean. It would be interesting to know if the geometric mean is closer to the harmonic mean or to the arithmetic mean. We show that it is closer to the harmonic mean by proving that

$$\sqrt{ab} - \frac{2ab}{a+b} \leq \frac{a+b}{2} - \sqrt{ab}.$$

Indeed, this is equivalent to

$$2\sqrt{ab} \leq \frac{2ab}{a+b} + \frac{a+b}{2},$$

which is just $2\sqrt{xy} \leq x + y$ for $x = \dfrac{2ab}{a+b}$ and $y = \dfrac{a+b}{2}$.

4.3. We have so far begged an important question. Is the harmonic mean of a and b always a mean? That is, is it always between a and b? Show that if $0 \leq a < b$, then $a < \dfrac{2ab}{a+b} < b$.

Solution. We have:

$$\begin{aligned} a &< b \\ a^2 &< ab, \ \text{(since } b > 0) \\ a^2 + ab &< 2ab, \\ a(a+b) &< 2ab, \\ a &< \frac{2ab}{a+b}, \end{aligned}$$

which is what we want. A similar argument will show that $\dfrac{2ab}{a+b} < b$. This shows that the harmonic mean is truly a mean.

As you solve this problem, you may find yourself writing this argument "backwards" from our solution above. Remember that the logic

of inequalities is not always "reversible". We must be sure we can start with something we know and end up with our result.

4.4. **a.** Look back at Figure 2.1, and connect O to Y as shown in Figure 4.1 below. Point Z be the foot of the perpendicular from X to OY. Using the same segment lengths as given in Problem 2.1 show that YZ is the harmonic mean of a and b. Deduce geometrically that the harmonic mean of a and b is less than or equal to their arithmetic mean.

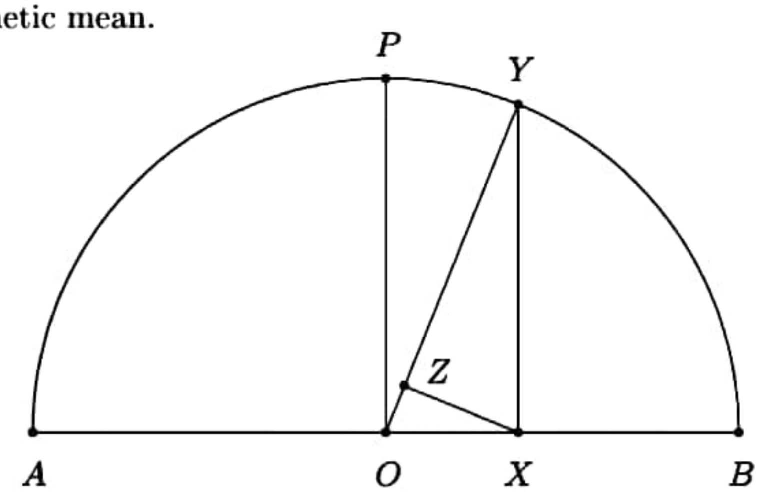

Figure 4.1

Solution. Triangles XYZ and OYX are similar, so $\dfrac{XY}{YZ} = \dfrac{OY}{XY}$. It follows that

$$YZ = \frac{XY^2}{OY} = \frac{\left(\sqrt{ab}\right)^2}{(a+b)/2} = \frac{2ab}{a+b},$$

which is just the harmonic mean of a and b.

Now since YZ is a leg of right triangle XYZ, while XY is its hypotenuse, we have $YZ \le XY$, or $\dfrac{2ab}{a+b} \le \dfrac{a+b}{2}$.

Equality holds if and only if triangle XYZ degenerates into a straight line. Then points X and O coincide, which means that $a = b$.

b. (This problem requires the 'triangle inequality', a geometric inequality which we haven't discussed in this volume.) Let h_a, h_b, h_c be the lengths of the altitudes of a triangle (with sides a, b, c). Prove that $2h_a$ is greater than the harmonic mean of h_b and h_c.

Solution. The key to this problem is the classic *triangle inequality*, and the connection between the altitudes of a triangle and its area. The triangle inequality says that the sum of any two sides of a triangle must be greater than the third side. So we know that $b + c > a$.

Now if K is the area of the triangle, then

$$K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c,$$

so we can rewrite the triangle inequality as:

$$\frac{2K}{h_b} + \frac{2K}{h_c} > \frac{2K}{h_a}.$$

Then $\dfrac{1}{h_b} + \dfrac{1}{h_c} > \dfrac{1}{h_a}$, which means that

$$\frac{h_b + h_c}{h_b h_c} > \frac{1}{h_a}.$$

Hence

$$2h_a > \frac{2h_b h_c}{h_b + h_c},$$

as desired.

4.5. Suppose we have two sets of n positive numbers each:

$$S_1 = \{a_1, a_2, a_3, \ldots, a_n\} \text{ and } S_2 = \{a_1, x, a_3, \ldots, a_n\}.$$

If $a_2 > x$, show that the harmonic mean of the numbers in S_1 is greater than the harmonic mean of those in S_2.

That is, replacing a number in a set by a smaller number decreases the harmonic mean of the set.

Solution. We write the solution for two sets of four numbers. The generalization is immediate.

We have:

$$a_2 > x;$$

$$\frac{1}{a_2} < \frac{1}{x};$$

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} < \frac{1}{a_1} + \frac{1}{x} + \frac{1}{a_3} + \frac{1}{a_4};$$

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}} > \frac{n}{\frac{1}{a_1} + \frac{1}{x} + \frac{1}{a_3} + \frac{1}{a_4}}.$$

This last inequality is the statement that the harmonic mean of S_1 is greater than or equal to that of S_2.

Note the various reversals of the direction of this inequality as we manipulate it.

4.6. Show that the harmonic mean of n numbers is between the largest and the smallest of them.

Solution. (This solution was suggested by Daniel Vitek.)

We use the notation $HM(S)$ to denote the harmonic mean of a set S of numbers.

Let $S_1 = \{a_1, a_2, a_3, \ldots, a_n\}$, where $a_1 \le a_2 \le a_3 \le \ldots \le a_n$. We will show that $HM(S_1) \ge a_1$.

Let $S_2 = \{a_1, a_1, a_3, \ldots, a_n\}$, where we have replaced a_2 by a_1 (which is smaller). Then, by Problem 4.3, we have $HM(S_1) \ge HM(S_2)$.

Let $S_3 = \{a_1, a_1, a_1, \ldots, a_n\}$, where we have replaced a_3 by a_1 (which is smaller). Then, by Problem 4.3 we have $HM(S_2) \geq HM(S_3)$, so $HM(S_1) \geq HM(S_3)$ as well.

Continuing in this way, we finally arrive at S_n, which consists of n copies of a_1. Then

$$HM(S_1) \geq HM(S_n) = \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_1} + \ldots + \dfrac{1}{a_1}} = \frac{n}{\left(\dfrac{n}{a_1}\right)} = a_1.$$

In the same way, we can prove that $HM(S_1) \leq a_n$.

4.7. If AM, GM, HM stand respectively for the arithmetic, geometric, and harmonic means of n positive numbers, show that $AM \geq GM \geq HM$. (We have already shown the first inequality in Chapter 3. Show the second.)

Solution. The inequality to be proven can be written as

$$\frac{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_1}}{n} \geq \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}},$$

which is just a special case of the AM-GM inequality for n quantities.

4.8. Given two points A and B on a line, find a point W on line segment AB such that $AW = WB$.

Solution. Point W is just the midpoint of line segment AB. There are other points W such that $AW = WB$, but they are not on segment AB.

4.9. Given two points A and B on a line, find a point X on line segment AB such that $BX = 2XA$.

Solution. We just trisect line segment AB, and choose as X the trisection point closer to A.

4.10. Given two points A and B on a line, find a point Y on line segment AB such that $AY = 2YB$.

Solution. Point Y is the other trisection point of line segment AB, the one closer to B.

4.11. Given two points A and B on a line, find another point Y' on line AB (distinct from the point Y in Problem 4.10) such that $AY' = 2Y'B$.

Solution. Take the point Y' such that $Y'B = BA$. Then $AY' = AB + BY' = 2Y'B$, as required. Note that the second trisection point of segment AB does not work.

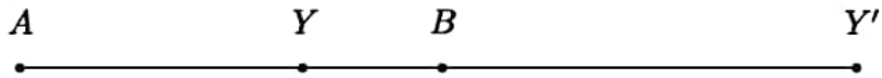

Figure 4.2

4.12. Are there any other points Z, anywhere in the plane, other than Y' (from Problem 4.11) such that $AZ = 2BZ$?

Solution. Yes. There are lots of them, not on line AB. It is a fascinating question, which we will not pursue here, where they all lie.

For a peek at this topic, look up *circles of Apollonius* in any advanced geometry book.

4.13. (Extra credit) Given two points A and B on a line, construct, with straightedge and compass, the points Y and Y' referred to in Problems 4.10 and 4.11.

Solution. A set of parallel lines cuts off proportional segments along any transversal.
Thus we have the following construction. Through point A we draw any line at all, different from AB. Along this line we want to create a set of four points in the ratios we require along line AB.
We can do this by marking off six equal segments, of any length, along this line, starting at point A: $AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5P_6$. Then $AP_2 : P_2P_3 = 2:1$ and $AP_6 : P_6P_3 = 2:1$ as well.
Next we join P_3 to point B, and draw a line through P_2 parallel to P_3B. This line will intersect AB at point Y inside segment AB such that $AY : YB = 2:1$.
Finally, we draw a line parallel to P_3B through point P_6. This line will intersect AB at point Y' outside segment AB such that $AY' : Y'B = 2:1$.

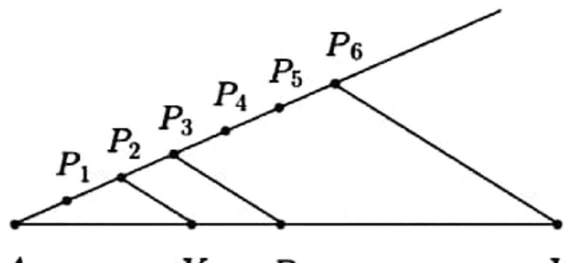

Figure 4.3

Given two points A and B on a line, if $AX : XB = AY : YB$, then points X and Y are called *harmonic conjugates* with respect to A and B. The points $\{A, X, B, Y\}$ are called a *harmonic range* on the line.

4.14. If X and Y are harmonic conjugates with respect to A and B, show that A and B are also harmonic conjugates with respect to X and Y.

Solution. We have $AX : XB = AY : YB$. We want $XA : YA = XB : YB$. These two proportions have the same cross products, so each implies the other.
We sometimes say that the second proportion is obtained from the first by "alternation of the means".

4.15. Generalize Problem 4.13. Given points A and B, and a point X between them, construct, with straightedge and compass, the point Y that is the harmonic conjugate of X. What if point X is given outside segment AB?

Solution. We can use a construction similar to that of Problem 4.13. We draw any line through B, and choose some point B' on this

line. We connect B to B', then draw a parallel to BB' through X, intersecting AB' at point X'. We now have $AX' : X'B' = AX : XB$. If we can construct Y' on AB' so that $AY' : Y'B' = AX' : X'B'$, we will have the required proportions along line AB', and can use parallel lines to locate point Y on AB with the required ratio.
Let us do this by drawing a third line through A, reversing the roles of X' and B' for new points along this third line. That is, we mark off a segment $AX'' = AX'$ along it. If we then mark off segment $B''X'' = B'X'$ along AB'' so that X'' is outside segment AB'' we have $AX'' : X''B'' = AX' : X'B' = AX : XB$. Drawing a line $X''Y'$ parallel to $B'B''$, with Y' on line AB', we have $AY' : Y'B' = AX'' : X''B'' = AX' : X'B' = AX : XB$.

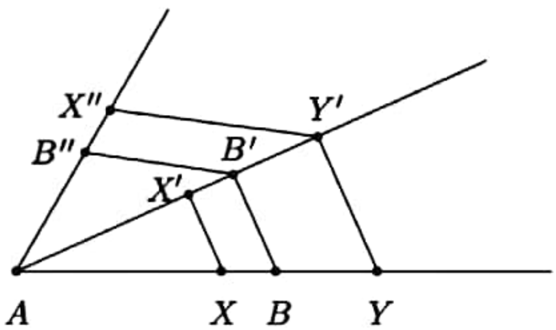

Figure 4.4

In turn, if we draw a line through Y' parallel to BB', it will intersect AB at point Y such that $AY : YB = AY' : Y'B' = AX : XB$, and Y is the harmonic conjugate of X with respect to segment AB.
If point X is outside segment AB, the construction proceeds analogously.

4.16. Let X and Y be harmonic conjugates with respect to A and B. Let $AX = a$, $AB = h$, and $AY = b$. Show that h is the harmonic mean of a and b.

Solution. We have

$$AX = a,\ XB = h - a,\ AY = b,\ YB = AY - AB = b - h,$$

and we have:

$$\begin{aligned} AX : XB &= AY : YB, \\ a : (h-a) &= b : (b-h), \\ ab - ah &= bh - ba, \\ 2ab &= h(a+b), \\ h &= \frac{2ab}{a+b}, \end{aligned}$$

which is the harmonic mean of a and b.

4.17. For two points A and B, where is the harmonic conjugate of the midpoint of AB?

Solution. This is an interesting question. Let M be the midpoint of AB. We need a point P such that $AM : MB = 1 = AP : PB$. But that means that $AP = PB$. This cannot happen, since $BP = AP - AB$, and AB is not zero. (We will not bother to talk about the situation when $AB = 0$. Then A and B will coincide, and there is no midpoint to talk about. Or, $AP = PB$ for any point on line AB.)
But notice that as point P recedes from segment AB, in either direction, the ratio $AP : PB$ approaches 1. We sometimes say that the harmonic conjugate of the midpoint M of AB is the *point at infinity* along line AB. This concept is made very clear in projective geometry. For now, it is just a figure of speech. But also see Problem 4.18.

4.18. If you ride your bike up a hill at 15 MPH, at what rate must you ride down the hill, so that your average speed for the two trips is 30 MPH? If you have trouble with this problem and Problem 4.17, you are in good company.
Albert Einstein is on record for finding the solution counterintuitive. But can you make a geometric diagram which shows what is happening?

Solution and discussion (for this problem and Problem 4.17)
Let's look at the motion problem first. The harmonic mean h of rates a and b is $\dfrac{2ab}{a+b}$. Here, we have $a = 15$, $h = 30$, and we are asked for b. So we need:

$$\begin{aligned}\frac{30b}{15+b} &= 30,\\ \frac{b}{15+b} &= 1,\\ \frac{15+b}{b} &= 1,\\ 15+b &= b,\\ 15 &= 0.\ (?!?!)\end{aligned}$$

So what's going on? It seems that there is no number b that works. In fact, if we do this more generally, and require the average of two rates to be double one of them, the algebra will again show no solution.
We can illustrate this situation geometrically. We know that if four points A, M, B, P form a harmonic range along line AB, then AB is the harmonic mean of AM and AP. We think of line AB as a number line, and let A have coordinate 0, M have coordinate 15, and B have coordinate 30. Then we want a point P so that the harmonic mean of AM and AP is AB. That will be asking for the harmonic conjugate of point M along line AB, and we know that this point does not exist.
Or, we know that it is the point at infinity. As Einstein wrote to a colleague, "Not until calculating did I notice that there is no time left for the way down!"

Chapter 5

Symmetry in Algebra, Part I

Symmetry is a fundamental mathematical concept. The study of symmetry, which is called *group theory*, has been a productive area of mathematical research for two centuries, and its treasury of uses and results shows no sign of being depleted.

In geometry, the symmetry in certain figures strikes the eye immediately, and the difficulty lies in harnessing it to achieve certain results. The same concept, in algebra, is more subtle. Algebraic symmetry appeals to the mind, not the eye, and reveals itself only slowly, as one works through a series of problems.

Example 5.1. Solve the following system of equations:

$$\begin{cases} x + 5y = 9 \\ 5x + y = 15. \end{cases}$$

Solution. Following the usual textbook solution, one would multiply one of the equations by 5, then subtract. This will of course get us the answer, and the method generalizes to any pair of simultaneous linear equations, and to simultaneous equations with more than two variables.

Or, we could solve one equation for x and substitute into the other equation. This method also generalizes for any pairs of simultaneous linear equations (although it gets difficult when we involve more variables).

But here's a more subtle way to solve this system, a way that generalizes in another direction: We have $x + 5y = 9$ and $5x + y = 15$. Adding, we find that $6x + 6y = 24$, so $x + y = 4$. Then we subtract this from the first equation to get $4y = 5$, and from the second to get $4x = 11$, and the solution is easy:

$$x = \frac{11}{4}, \ y = \frac{5}{4}.$$

Why does this method work? Neither equation is symmetric in x and y on its own, but as a system, there is symmetry: the two variables play the same roles in the system. In other words, if we looked only at the left sides of the equations, and interchanged x and y, we would not know the difference. And in fact it is the *form* of the left sides of the equations, not the particular *numbers* on the right, that dictates the algebraic procedures we use to solve them.

The following problems can be thought of as generalizations of this first simple example. In general, if we perceive algebraic symmetry in a system of equations, it is helpful to act on them so as to preserve this symmetry.

Problems

5.1. Solve simultaneously $\begin{cases} x+2y+z=14 \\ 2x+y+z=12 \\ x+y+2z=18 \end{cases}$

5.2. Solve simultaneously $\begin{cases} x+y=7 \\ y+z=-2 \\ z+x=9 \end{cases}$

5.3. Solve simultaneously $\begin{cases} xy=6 \\ yz=15 \\ zx=10 \end{cases}$

5.4. Solve simultaneously $\begin{cases} (x+1)(y+1)=24 \\ (y+1)(z+1)=30 \\ (z+1)(x+1)=20 \end{cases}$

5.5. Solve simultaneously $\begin{cases} xy-x-y=11 \\ yz-y-z=14 \\ zx-z-x=19 \end{cases}$

5.6. Solve simultaneously $\begin{cases} x(x+y+z)=4 \\ y(x+y+z)=6 \\ z(x+y+z)=54 \end{cases}$

5.7. Solve simultaneously $\begin{cases} x+[y]+\{z\}=1.1 \\ \{x\}+y+[z]=2.2 \\ [x]+\{y\}+z=3.3 \end{cases}$

In this problem, the notation $[x]$ means "the greatest integer not exceeding x", and $\{x\}$ means "the fractional part of x", that is, $\{x\} = x-[x]$. So, for example, $[5.2]=5$ and $\{5.2\}=0.2$, while $[7]=7$ and $\{7\}=0$.

5.8. If a is a fixed positive real number, solve simultaneously:

$$\begin{cases} x^2-xy=a \\ y^2-xy=a(a-1). \end{cases}$$

5.9. Solve the following system of n equations in n unknowns (where n is some integer greater than 2):

$$\begin{cases} x_2+x_3+x_4+\ldots+x_n=1 \\ x_1+x_3+x_4+\ldots+x_n=2 \\ x_1+x_2+x_4+\ldots+x_n=3 \\ \ldots \\ x_1+x_2+x_3+\ldots+x_{n-1}=n. \end{cases}$$

Problems

5.12. Show that the geometric mean of the positive numbers a, b lies between their two values.

5.13. Generalize Example 5.2 and Problem 5.12 for n positive numbers $a_1, a_2, \ldots, a_n$.

5.14. Prove that for any real numbers a, b, c,

$$7\sqrt{a^2+b^2+c^2} \le \sqrt{3}\max\{|-2a+3b+6c|, |6a-2b+3c|, |3a+6b-2c|\}.$$

5.15. Let a, b, c, x, y, z be real numbers such that

$$4x+y=b+4c,\ 4y+z=c+4a,\ 4z+x=a+4b.$$

Prove that

$$x^2+y^2+z^2 \ge ab+bc+ca.$$

5.16. Prove that for any real numbers a, b, c,

$$a^2+b^2+c^2-ab-bc-ca \ge \frac{3}{4}(a-b)^2.$$

5.17. (Dominik Teiml, https://www.awesomemath.org/wp-pdf-files/math-reflections/mr-2015-05/mr_4_2015_solutions_2.pdf, accessed June 2016) Find the maximum possible value of k for which

$$\frac{a^2+b^2+c^2}{3} - \left(\frac{a+b+c}{3}\right)^2 \ge k\cdot\max\{(a-b)^2, (b-c)^2, (c-a)^2\},$$

for all real numbers a, b, c.

5.18. In triangle ABC, $\dfrac{\pi}{7} < A \le B \le C < \dfrac{5\pi}{7}$. Prove that

$$\sin\frac{7A}{4} - \sin\frac{7B}{4} + \sin\frac{7C}{4} > \cos\frac{7A}{4} - \cos\frac{7B}{4} + \cos\frac{7B}{4}.$$

5.19. In triangle ABC, $\max\{A, B, C\} < 120°$. Prove that

$$\sin A - \sin B + \sin C < \sqrt{3}(\cos A - \cos B + \cos C).$$

5.20. In any triangle ABC, show that

$$\cos\frac{A}{2} + \cos\frac{B}{2} > \cos\frac{C}{2}.$$

Solutions

5.1. Solve simultaneously $\begin{cases} x+2y+z=14 \\ 2x+y+z=12 \\ x+y+2z=18 \end{cases}$

Solution. Adding the three given equations, we get

$$4x+4y+4z=44,$$

or

$$x+y+z=11.$$

If we subtract this equation from each of the given equations in turn, we find very quickly that $y = 3$, $x = 1$, and $z = 7$. The reader who has tried substitution will appreciate how much easier our solution is.

5.2. Solve simultaneously $\begin{cases} x + y = 7 \\ y + z = -2 \\ z + x = 9 \end{cases}$

Solution. Adding the three equations, we find

$$2x + 2y + 2z = 14,$$

or

$$x + y + z = 7.$$

Subtracting each of the given equations in turn from this one, we find $z = 0$, $x = 9$, and $y = -2$.

5.3. Solve simultaneously $\begin{cases} xy = 6 \\ yz = 15 \\ zx = 10 \end{cases}$

Solution. Taking a hint from Problem 5.2, we multiply the three equations together to find

$$x^2y^2z^2 = 6 \cdot 15 \cdot 10,$$

so $xyz = \pm 30$. Then we divide this equation by each of the given equations to find

$$(x, y, z) = \pm(2, 3, 5).$$

5.4. Solve simultaneously $\begin{cases} (x+1)(y+1) = 24 \\ (y+1)(z+1) = 30 \\ (z+1)(x+1) = 20 \end{cases}$

Solution. Let's be quick about this. Let $p = x + 1$, $q = y + 1$, and $r = z + 1$. Then we have $pq = 24$, $qr = 30$, and $rp = 20$, and we have the same kind of equation as in Problem 5.3. We find that $(p, q, r) = (4, 6, 5)$ or $(-4, -6, -5)$. The corresponding values for (x, y, z) are $(3, 5, 4)$ and $(-5, -7, -6)$.

5.5. Solve simultaneously $\begin{cases} xy - x - y = 11 \\ yz - y - z = 14 \\ zx - z - x = 19 \end{cases}$

Solution. We can make this problem resemble Problem 5.4 by adding 1 to both sides of each equation. For example, the first equation becomes

$$xy - x - y + 1 = 12,$$

or

$$(x - 1)(y - 1) = 12.$$

We then let $p = x - 1$, $q = y - 1$, $r = z - 1$ and proceed as before. We find that $(x, y, z) = (5, 4, 6)$ or $(-3, -2, -4)$.

5.6. Solve simultaneously $\begin{cases} x(x+y+z)=4 \\ y(x+y+z)=6 \\ z(x+y+z)=54 \end{cases}$

Solution. Adding all three equations, and factoring the left side, we find that

$$(x+y+z)^2 = 64,$$

so

$$x+y+z = \pm 8.$$

Then we divide each equation by this relation, to find

$$(x,y,z) = \left(\frac{1}{2}, \frac{3}{4}, \frac{27}{4}\right) \text{ or } \left(-\frac{1}{2}, -\frac{3}{4}, -\frac{27}{4}\right).$$

5.7. Solve simultaneously $\begin{cases} x+[y]+\{z\}=1.1 \\ \{x\}+y+[z]=2.2 \\ [x]+\{y\}+z=3.3 \end{cases}$

In this problem, the notation $[x]$ means "the greatest integer not exceeding x", and $\{x\}$ means "the fractional part of x", that is, $\{x\} = x - [x]$. So, for example, $[5.2] = 5$ and $\{5.2\} = 0.2$, while $[7] = 7$ and $\{7\} = 0$.

Solution. Adding the given equations, we get $2x + 2y + 2z = 6.6$, or

$$x+y+z = 3.3. \tag{5.1}$$

The key to the rest of the solution is that for any a, we have $a = [a] + \{a\}$. Subtracting the first of the given equations from (5.1) gives us

$$\{y\} + [z] = 2.2,$$

which means that $[z] = 2$ and $\{y\} = 0.2$. Subtracting the third of the given equations from (5.1), we find that

$$\{x\} + [y] = 0,$$

which shows that $\{x\} = 0$ and $[y] = 0$ as well. Hence x is an integer. From these two steps we can conclude that $y = 0.2$. Subtracting the second of the given equations from (5.1), we find that

$$[x] + \{z\} = 1.1,$$

so $[x] = 1$ and $\{z\} = 0.1$, and from the previous relations, we can infer that $x = 1$ and $z = 2.1$.

5.8. If a is a fixed positive real number, solve simultaneously:

$$\begin{cases} x^2 - xy = a \\ y^2 - xy = a(a-1). \end{cases}$$

Solution. Adding the equations, we find

$$x^2 - 2xy + y^2 = (x-y)^2 = a^2,$$

We can use the technique of Problem 5.18. Consider the triangle $A'B'C'$ with $A' = 120° - A$, $B' = 120° - B$ and $C' = 120° - C$. Indeed, there is such a triangle $A'B'C'$, since $A', B', C' > 0$ and

$$A' + B' + C' = 3 \cdot 120° - 180° = 180°.$$

Let $a' = B'C'$, $b' = C'A'$, $c' = A'B'$ and let R' be the circumradius of triangle $A'B'C'$. From the triangle inequality, we have $b' < a' + c'$ and so, using the Extended Law of Sines, $2R'' \sin B' < 2R'' \sin A' + 2R'' \sin C'$. It follows that

$$\sin(120° - B) < \sin(120° - A) + \sin(120° - C),$$

implying

$$\frac{\sqrt{3}}{2}\cos B - \frac{1}{2}\sin B < \frac{\sqrt{3}}{2}\cos A - \frac{1}{2}\sin A + \frac{\sqrt{3}}{2}\cos C - \frac{1}{2}\sin C.$$

Hence the conclusion.

5.20. In any triangle ABC, show that

$$\cos\frac{A}{2} + \cos\frac{B}{2} > \cos\frac{C}{2}.$$

Solution. We use the ideas of the previous problem. We want to construct a triangle with angles

$$A' = 90° - \frac{A}{2},\ B' = 90° - \frac{B}{2},\ C' = 90° - \frac{C}{2}.$$

Because $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90°$, there is in fact such a triangle. Following the reasoning of the previous solution, we find that

$$\sin\left(90° - \frac{A}{2}\right) + \sin\left(90° - \frac{B}{2}\right) > \sin\left(90° - \frac{C}{2}\right),$$

which is equivalent to the result we want.

Chapter 6

Symmetry in Algebra, Part II

The topic of algebraic symmetry is central to a study of many aspects of algebra. For this reason, we continue our discussion of it. We will come back to our central theme of inequalities in a later chapter.

We start with a simple quadratic equation. We know how to solve quadratic equations by factoring.

Example 6.1. Solve $x^2 - 5x + 6 = 0$.

Solution.

$$\begin{aligned}
x^2 - 5x + 6 &= 0\\
(x-3)(x-2) &= 0\\
x = 3 \text{ or } x &= 2.
\end{aligned}$$

So if $x - 2$ is a factor of the original polynomial, then $x = 2$ is a root of the associated polynomial equation. In fact, this will always work. There is nothing special about the number 2 or the factor $x - 2$:

It is not hard to see that the following statement is true in general:

Theorem. *If $x - a$ is a factor of the quadratic polynomial $P(x)$, then*

$$P(a) = 0. \tag{6.1}$$

Is the converse of this statement true? Let us look at another example:

Example 6.2. Solve $6x^2 - x - 1 = 0$.

Solution.

$$\begin{aligned}
6x^2 - x - 1 &= 0\\
(2x-1)(3x+1) &= 0\\
2x - 1 = 0 \text{ or } 3x + 1 &= 0\\
x = \frac{1}{2} \text{ or } x &= -\frac{1}{3}.
\end{aligned}$$

We see that $x = \dfrac{1}{2}$ is a root of the equation we started with, but $x - \dfrac{1}{2}$ is not a factor. Or is it? If we had a bit more fondness for fractions, we could

have rewritten the original equation as:

$$x^2 - \frac{x}{6} - \frac{1}{6} = 0$$

$$\left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right) = 0$$

$$x = \frac{1}{2} \text{ or } x = -\frac{1}{3}.$$

Why didn't we do this in the first place? Because factoring takes some guesswork, and it's easier for us to guess about integers than about rational numbers. But this is our failing, and not that of the equation we are solving. Considering the second solution of our equation, we can see that it is true that if $x = \frac{1}{2}$ is a root, then $x - \frac{1}{2}$ is a factor.

Indeed, the whole story is made simpler if we consider only quadratic equations whose lead coefficient (the coefficient of x^2) is 1. We will do so for the remainder of this chapter, and now the converse of statement (6.1) is in fact true. We can state the very interesting *Factor Theorem* for quadratic

♦

Example 6.3. If α and β are the roots of $x^2 - px + q = 0$, find the value of $\alpha^3 + \beta^3$ (in terms of p and q).

Solution 1. By direct calculation (or applying the binomial theorem, if you are familiar with it), we have:

$$(\alpha + \beta)^3 = \alpha^2 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta),$$

so

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = p^3 - 3pq.$$

Solution 2. Since α and β are both solutions to the equation $x^2 - px + q = 0$. they must also satisfy $x^3 - px^2 + qx = 0$. So we have

$$\alpha^3 - p\alpha^2 + q\alpha = 0,$$
$$\beta^3 - p\beta^2 + q\beta = 0.$$

Adding, we have

$$\alpha^3 + \beta^3 - p(\alpha^2 + \beta^2) + q(\alpha + \beta) = 0,$$

or (from the result of Problem 6.4),

$$\alpha^3 + \beta^3 - p(p^2 - 2q) + pq = 0.$$

Hence

$$\alpha^3 + \beta^3 = p^3 - 2pq - pq = p^3 - 3pq,$$

as before.

Example 6.4. Use the result of Example 6.3 to find the value of $\alpha^4 + \beta^4$ (in terms of p and q).

Solution. Let S_4 be the expression in p and q that represents $\alpha^4 + \beta^4$. Let S_1, S_2, S_3 represent the corresponding expressions for $\alpha+\beta$, $\alpha^2+\beta^2$, $\alpha^3+\beta^3$, respectively. We already know that:

$$S_1 = \alpha + \beta = p$$
$$S_2 = \alpha^2 + \beta^2 = p^2 - 2q$$
$$S_3 = \alpha^3 + \beta^3 = p^3 - 3pq.$$

We use the idea of solution 2 from Example 6.3. The numbers α and β must satisfy

$$\alpha^4 - p\alpha^3 + q\alpha^2 = 0$$
$$\beta^4 - p\beta^3 + q\beta^2 = 0.$$

Adding, we have

$$S_4 - pS_3 + qs_2 = 0 \quad \text{or} \quad S_4 = pS_3 - qS_2.$$

Substituting the values we already know, we have

$$S_4 = p(p^3 - 3pq) - q(p^2 - 2q) = p^4 - 3p^2q - p^2q + 2q^2 = p^4 - 4p^2q + 2q^2.$$

6.6. What value should we assign to S_0? For S_1? Show that equation (6.1) remains valid for $n = 2$.

Solution. If $n = 0$, then we can write $S_0 = \alpha^0 + \beta^0 = 2$, no matter what equation we are talking about. If $n = 1$, then we can write

$$S_1 = \alpha^1 + \beta^1 = p.$$

Let us look at equation (6.1) for the case $n = 2$.
It asserts that $S_2 = pS_1 - qS_0$, or $\alpha^2 + \beta^2 = p \cdot p - q \cdot 2$, which we already know is true from Problem 6.4.

6.7. Does equation (6.1) hold for $n = 1$? For $n = 0$? For $n = \frac{3}{2}$? Note that a proof by mathematical induction does not hold for these values of n.

Solution. For $n = 1$ we have $S_1 = pS_0 - qS_{-1}$, or

$$\alpha + \beta = 2p - q\left(\frac{1}{\alpha} + \frac{1}{\beta}\right). \tag{6.4}$$

We recall that $q = \alpha\beta$ and $p = \alpha + \beta$, so that we can rewrite (6.4) as:

$$\alpha + \beta = 2p - (\beta + \alpha) = 2(\alpha + \beta) - (\beta + \alpha),$$

which we can easily see is true.

For $n = 0$, we have $S_0 = pS_{-1} - qS_{-2}$, or

$$2 = (\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - \alpha\beta\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right), \tag{6.5}$$

and we have a computation on our hands. We have:

$$(\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{\alpha + \beta}{\alpha} + \frac{\alpha + \beta}{\beta} = 1 + \frac{\beta}{\alpha} + 1 + \frac{\alpha}{\beta},$$

$$\alpha\beta\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) = \frac{\beta}{\alpha} + \frac{\alpha}{\beta}.$$

Combining these two, we find that the right-hand side of (6.5) is indeed equal to 2. Equation (6.2) is true for $n = 0$.

Finally, for $n = \frac{3}{2}$, we have $S_{\frac{3}{2}} = pS_{\frac{1}{2}} - qS_{-\frac{1}{2}}$, or

$$\alpha^{\frac{3}{2}} + \beta^{\frac{3}{2}} = p\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right) - q\left(\frac{1}{\alpha^{\frac{1}{2}}} + \frac{1}{\beta^{\frac{1}{2}}}\right).$$

The first term on the right is equal to

$$(\alpha + \beta)\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right) = \alpha^{\frac{3}{2}} + \beta^{\frac{3}{2}} + \alpha^{\frac{1}{2}}\beta + \beta^{\frac{1}{2}}\alpha.$$

The second term is equal to

$$\alpha\beta\left(\frac{1}{\alpha^{\frac{1}{2}}} + \frac{1}{\beta^{\frac{1}{2}}}\right) = \alpha^{\frac{1}{2}}\beta + \beta^{\frac{1}{2}}\alpha.$$

It follows that

$$\begin{aligned} \alpha+\beta+\gamma &= p \\ \alpha\beta+\alpha\gamma+\beta\gamma &= q \\ \alpha\beta\gamma &= r. \end{aligned}$$

For Problems 6.11 through 6.15, if α, β, and γ are the roots of the equation

$$x^3 - px^2 + qx - r = 0, \tag{6.6}$$

express in terms of p, q, and r the value of:

6.11. $\alpha^2+\beta^2+\gamma^2$.

Solution. Following Problem 6.3, Method II, we find that

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma),$$

or

$$p^2 = \alpha^2+\beta^2+\gamma^2+2q.$$

It follows that

$$\alpha^2+\beta^2+\gamma^2 = p^2-2q.$$

Compare this result with the result of Problem 6.4. How does it generalize?

6.12. $\alpha^3+\beta^3+\gamma^3$.

Solution. We again follow Problem 6.3, Method II:

$$\begin{aligned} \alpha^3 - p\alpha^2 + q\alpha - r &= 0 \\ \beta^3 - p\beta^2 + q\beta - r &= 0 \\ \gamma^3 - p\gamma^2 + q\gamma - r &= 0. \end{aligned}$$

Adding, we have

$$\begin{aligned} \alpha^3+\beta^3+\gamma^3 &= p(\alpha^2+\beta^2+\gamma^2) - q(\alpha+\beta+\gamma) + 3r \\ &= p(p^2-2q) - pq + 3r \\ &= p^3 - 3pq + 3r. \end{aligned}$$

6.13. $\alpha^4+\beta^4+\gamma^4$.

Solution. We can "bootstrap" this computation, as we did in Problem 6.3, Method II, to find the sums of higher powers of α, β and γ. In fact, we have been beaten to this idea by none other than Sir Isaac Newton, who is credited with discovering a general formula for the sums of powers of roots of a polynomial equation.

Suppose α, β, γ are the roots of

$$x^3 - px^2 + qx - r = 0.$$

It follows that

$$\alpha+\beta+\gamma=p$$
$$\alpha\beta+\alpha\gamma+\beta\gamma=q$$
$$\alpha\beta\gamma=r.$$

For Problems 6.11 through 6.15, if α, β, and γ are the roots of the equation

$$x^3-px^2+qx-r=0, \tag{6.6}$$

express in terms of p, q, and r the value of:

6.11. $\alpha^2+\beta^2+\gamma^2$.

Solution. Following Problem 6.3, Method II, we find that

$$(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma),$$

or

$$p^2=\alpha^2+\beta^2+\gamma^2+2q.$$

It follows that

$$\alpha^2+\beta^2+\gamma^2=p^2-2q.$$

Compare this result with the result of Problem 6.4.
How does it generalize?

6.12. $\alpha^3+\beta^3+\gamma^3$.

Solution. We again follow Problem 6.3, Method II:

$$\alpha^3-p\alpha^2+q\alpha-r=0$$
$$\beta^3-p\beta^2+q\beta-r=0$$
$$\gamma^3-p\gamma^2+q\gamma-r=0.$$

Adding, we have

$$\begin{aligned}\alpha^3+\beta^3+\gamma^3&=p(\alpha^2+\beta^2+\gamma^2)-q(\alpha+\beta+\gamma)+3r\\&=p(p^2-2q)-pq+3r\\&=p^3-3pq+3r.\end{aligned}$$

6.13. $\alpha^4+\beta^4+\gamma^4$.

Solution. We can "bootstrap" this computation, as we did in Problem 6.3, Method II, to find the sums of higher powers of α, β and γ. In fact, we have been beaten to this idea by none other than Sir Isaac Newton, who is credited with discovering a general formula for the sums of powers of roots of a polynomial equation.

Suppose α,β,γ are the roots of

$$x^3-px^2+qx-r=0.$$

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Then $x^4 - px^3 + qx^2 - rx = 0$ as well, so

$$\alpha^4 - p\alpha^3 + q\alpha^2 - r\alpha = 0$$
$$\beta^4 - p\beta^3 + q\beta^2 - r\beta = 0$$
$$\gamma^4 - p\gamma^3 + q\gamma^2 - r\gamma = 0.$$

Adding, we obtain

$$\alpha^4 + \beta^4 + \gamma^4 - p(\alpha^3 + \beta^3 + \gamma^3) + q(\alpha^2 + \beta^2 + \gamma^2) - r(\alpha + \beta + \gamma) = 0,$$

or

$$\alpha^4 + \beta^4 + \gamma^4 - p(p^3 - 3pq + 3r) + q(p^2 - 2q) - rp = 0.$$

This gives us the required expression:

$$\alpha^4 + \beta^4 + \gamma^4 = p^4 - 4p^2q + 4pr + 2q^2.$$

6.14. $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$. (Assume that $\alpha, \beta, \gamma \neq 0$.)

Solution. The given expression equals

$$\frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}.$$

The denominator of this fraction is clearly r^2. The numerator looks like it is related to q^2. And indeed, we have

$$q^2 = (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) + 2\alpha\beta\gamma(\alpha + \beta + \gamma).$$

Using this information, we quickly find that the required expression equals

$$\frac{q^2 - 2pr}{r^2}.$$

6.15. $\dfrac{\alpha\beta}{\gamma} + \dfrac{\alpha\gamma}{\beta} + \dfrac{\beta\gamma}{\alpha}$. (Assume that $\alpha, \beta, \gamma \neq 0$.)

Solution. The given expression equals

$$\frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{\alpha\beta\gamma}.$$

Using the computation from Problem 6.14, we find that this equals

$$\frac{q^2 - 2pr}{r}.$$

6.16. Suppose α, β, and γ are roots of equation (6.3), and suppose S_n is an expression in p, q, and r equal to the sum $\alpha^n + \beta^n + \gamma^n$, where n is a natural number. Find an expression for S_n in terms of S_{n-1}, S_{n-2}, and S_{n-3}. This will generalize the "bootstrapping" operation described in the solution to Example 6.4.

Solution. As in Example 6.4, we know that α, β, γ are solutions to the equation

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} = 0,$$